if it were the main text. That would be like using a book of cooking recipes as the main text for organic chemistry. If we mathematicians abandon the goal of logical development, who will replace us?

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Retarded dynamical systems: stability and characteristic functions, by G. Stépán. Longman Scientific & Technical, Pitman Research Notes in Mathematics Series, no. 210, 1989, 151 pp; $32.00. ISBN 0-582-03932-0

In engineering, biology, and physics, one often encounters dynamical systems that may be described a systems with memory, or hereditary systems, or systems with delayed feedback or time lag. The mathematical formulation and basic theory of the differential equations that describe such systems may be said to have begun with the work of A. D. Myshkis [9] and since that time there has been a growing body of mathematical research on many aspects of theory and applications, and the development of a general theory for what are now called functional differential equations (FDEs). The present book is devoted to an aspect of great practical importance, the stability theory for these equations. As will be explained below, the local stability theory depends on analysis of the location of zeros of associated "characteristic functions." A variety of methods have been proposed for treating the stability problem. Among these is the method of Pontryagin, described in [10] and in the book of Bellman and Cooke [1], but it is complicated for equations with more than one delay. Another is the D-subdivision or D-partition method, in which the space of the parameters of the equation is divided by hypersurfaces, the points of which correspond to quasipolynomials having at least one zero on the imaginary axis. This method, and others such as the tau-decomposition method and Nyquist criterion are described thoroughly in the books of El'sgol'ts and Norkin [3], MacDonald [8], and Kolmanovskii and Nosov [6]. Besides these, Liapunov functional techniques (see Hale [4], Yoshizawa [12]) are sometimes useful for either linear or nonlinear problems. Some general results for one delay equations are given in Cooke and van den Driessche [2]. (G. Boese has pointed out that hypothesis (iv) in Theorem 1 must be strengthened in the general case.)

Stépán comments, with considerable justification, that "none of these methods can be used generally for functional differential equations." For instance, the widely-used D-subdivision method depends heavily upon the knowledge of the hypersurfaces, which is generally difficult to find. His book is devoted to this problem, and consists of two main parts. The first part is the explanation of his own method, which he calls the "direct stability investigation," and which is presented with full proofs in Chapter 2. The second part is the construction of so-called stability charts, carried out for many equations with the aid of his direct method in Chapters 3 and 4. In sum, these provide very useful and quite broadly applicable tools for handling the stability problem.

In order to explain the contributions in this book in more detail, we begin with some basic definitions. A retarded functional differential equation (RFDE) describes a system in which the rate of change of state is determined by the present and past states. We may formulate such an equation as follows. Let $h \geq 0$ be a given number (or $+\infty$), let $\mathbb{R}^n$ be n-dimensional real space with norm $|\cdot|$, and let B denote the Banach space of continuous functions on $[-h, 0]$ into $\mathbb{R}^n$ with the norm

$$\|\phi\| = \sup_{\theta\in[-h,0]} |\phi(\theta)|, \qquad \phi \in B.$$

Then an RFDE is an equation of the form

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $f: R \times B \to \mathbb{R}^n$ and x_t in B is defined by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-h, 0].$$

h is called the lag or delay, and the equation is said to have bounded or unbounded delay according to whether h is finite or ∞. A function $x: \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of (1) with the initial condition $x_\sigma = \phi$, $\sigma \in \mathbb{R}$, $\phi \in B$, if there exists a scalar $\delta > 0$ such that $x_t \in B$, $x_\sigma = \phi$, and $x(t)$ satisfies (1) for t in $[\sigma, \sigma + \delta]$.

If the rate of change also depends on past values of the rate of change, the describing equation is called a neutral functional differential equation (NFDE). Although the book treats these equations too, we shall for the purposes of this review restrict attention to RFDEs.

It is not difficult to define the basic concepts of Liapunov stability and asymptotic stability of an equilibrium (constant) solution of (1), in analogy with definitions for ordinary differential equations. These, and fundamental ideas about Liapunov functionals, are described in Krasovskiĭ [7], Hale [4], and other authors. In treating the stability of constant solutions, one of the basic methods is to linearize around the equilibrium, thus obtaining a linear RFDE of the form

$$\dot{x}(t) = L(x_t) = \int_{-\infty}^{0} [d\eta(\theta)]x(t+\theta) \tag{2}$$

where L is a continuous linear functional and η is an $n \times n$ matrix of functions of bounded variation on $(-\infty, 0]$. Associated with (2) is the characteristic function given by

$$D(\lambda) = \det\left(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta}\, d\eta(\theta)\right), \qquad \lambda \in C. \tag{3}$$

A function of this form is an exponential polynomial or a more general function. If η is constant in $(-\infty, -h)$, the integral becomes a finite integral, and the equation has a bounded delay. If the kernel η is constant except for a finite number of finite jumps, the equation is frequently called a differential-difference equation. The function D is called stable by the author if every solution of the equation $D(\lambda) = 0$ has negative real part.

For ordinary differential equations, the stability of the characteristic polynomial is equivalent to (local) exponential asymptotic stability of the trivial solution. For FDEs, the situation is more complicated, but the author shows that under the condition that there exists $\nu > 0$ such that

$$(4) \qquad \int_{-\infty}^{0} e^{-\nu\theta} |d\eta_{jk}(\theta)| < +\infty, \qquad j, k = 1, \ldots, n$$

stability of the characteristic function is necessary and sufficient for exponential asymptotic stability of the RFDE (2). Therefore, the body of the book is devoted to the problem of stability of the characteristic function.

The direct stability investigation begins with the well-known fact that the number of zeros of $D(\lambda)$ in the right-half plane, given that there are no zeros on the imaginary axis, is given by

$$N = \frac{1}{2\pi i} \lim_{H \to +\infty} \oint D'(\lambda)/D(\lambda)\, d\lambda$$

where the integration is over a "Bromwich contour" composed of a semicircle of radius H in the right-half plane and a segment of the imaginary axis. Now let $R(\omega)$ and $S(\omega)$ denote the real and imaginary parts, respectively, of $D(i\omega)$ for $\omega \in [0, \infty)$. The following theorem is proved:

Theorem 2.15. *Let the dimension n of the RFDE (2) be even, $n = 2m$. If $D(\lambda)$ has no zero on the imaginary axis, and (4) holds, then the number N of zeros with positive real parts is given by*

$$N = m + (-1)^m \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k)$$

where the summation is over the finite number of real positive zeros of R, $\rho_1 \geq \cdots \rho_r > 0$. If n is odd, then a similar formula is obtained for N in terms of the sign of R evaluated at the finite number of nonnegative zeros of S.

The advantage of Stépán's result is in part its generality—it holds for equations with multiple discrete delays and for distributed delays. And secondly, explicit formulas are given, which involve the presumably simpler problem of finding the real zeros of R and S. By setting $N = 0$, necessary and sufficient conditions for asymptotic stability are obtained and these are stated as Theorem 2.19. The author also derives analogous results for certain classes of NFDEs, as well as several necessary conditions and several sufficient conditions for stability of RFDEs.

A stability chart is a diagram in the parameter space (in practice, a plane or three-dimensional space) which shows those regions in which the equilibrium of a system is stable. In applications, it is these diagrams which are often of foremost importance, since they show what sets of parameters may give rise to oscillations or instability. Stépán comments: "They are useful guides for engineers in design work. Moreover, ... have an important contribution in understanding the often peculiar physical behaviour of retarded dynamical systems." In the first section of Chapter 3, the author uses his Theorem 2.19 to obtain necessary and sufficient conditions for exponential asymptotic stability for scalar nth order equations with either all even or all odd order derivatives and with a single delay. The conditions consist of sets of inequalities relating the coefficients; the author uses these to construct stability charts in several cases. Other examples include an equation with two discrete delays. The structure of the stability charts is intriguing, usually consisting of many disjoint sets. Have a look at Figure 3.13, a chart for the equation with two delays, drawn in the space of these delays. The author does not attempt to draw general conclusions about the structure of these charts for multiple delay equations. Some examples of equations with continuous (distributed) delay, and with unbounded delay are also treated.

Chapter 4, entitled *Applications*, begins with a section on Lotka-Volterra type predator-prey systems, including one with a distributed delay in the conversion of prey into predator. Asymptotic stability for the linearization around an equilibrium is treated for several choices of the kernel. For one choice of the kernel, the existence of a Hopf bifurcation is shown, and it is proved that the bifurcation is supercritical, that is, the bifurcating periodic solution is stable when the delay is just above a critical value. The proof is based on the algorithm of Hasssard, Kazarinoff, and Wan [5]. See the papers of Stech [11] for recent work on Hopf bifurcation for FDEs. The chapter also contains interesting examples of man-machine control systems, robotics, and machine tool vibrations, in all of which there may be significant effects due to delays.

The book appears to be carefully written and quite free of errors or misprints. The author has pointed out to the reviewer that there is an error in the reformulation of a problem as an abstract differential equation (4.18), since differentiability even of the initial function is required in defining the operator A. Near the bottom of p. 16, there is an obvious misprint, where ϕ should be replaced by ζ.

The book is quite self-contained and its organization is clear. It provides some very useful methods for analyzing the question of stability for FDEs. Anyone with a need for such tools should have this text at hand.

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Classification theories of polarized varieties, by Takao Fujita. Cambridge University Press, 1990, 205 pp., $64.50. ISBN 0-521-39202-0

For simplicity we work over the complex numbers, $\mathbb{C}$, in this review. Let X be a projective variety, i.e. a reduced, algebraic subset of some projection space $\mathbb{P}^N$. Set theoretically X is defined by homogeneous polynomials. Assume moreover that X is irreducible and n-dimensional.

A natural way to try to understand X is to study a *hyperplane section*, H, of X, i.e. the intersection, $H = X \cap \mathbb{P}^{N-1}$, of X with a linear hyperplane, $\mathbb{P}^{N-1} \subset \mathbb{P}^N$, of $\mathbb{P}^N$. The hope here is that the hyperplane section is simpler than X and still contains usable information about X. This notion is quite simple—after a linear change of coordinates it is nothing more than setting one of the variables of the defining polynomials equal to 0.

Let me give some examples. Assume that the intersection is transverse and H is smooth. One of the simplest possible manifolds, H, is $\mathbb{P}^{n-1}$. If $n = \dim X = 1$, then H would be a single point and it is not too difficult to show that X is isomorphic to $\mathbb{P}^1$, $X \cong \mathbb{P}^1$. For $n = 2$, it is easy to see there are quite a few examples. One is $X \cong \mathbb{P}^2$ and H a linear $\mathbb{P}^1$ on $\mathbb{P}^2$. A second is $X \cong \mathbb{P}^2$ and H equal to a smooth conic, i.e. H is the zero set of an irreducible homogeneous polynomial of degree 2 on $\mathbb{P}^2$. A third example is X, a smooth hypersurface of degree 2 in $\mathbb{P}^3$ and H the intersection with a linear $\mathbb{P}^2 \subset \mathbb{P}^3$ that is transverse to X. Note that in this case $X = \mathbb{P}^1 \times \mathbb{P}^1$ and H can be taken to be the diagonal. Though there are many others, the very beautiful fact is that for all other examples, (X, H), with X a smooth surface, X is a Hirzebruch surface, F_r, i.e. X is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, and H is equal to a section. The Hirzebruch surfaces and their hyperplane sections are very well understood, (see [Ha, Chapter V, §2]). There is one for each integer $r \geq 0$ with $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and r the smallest integer such that there exists a section E of F_r over $\mathbb{P}^1$ with $E^2 = -r$. After this escalation of complication for $n = 2$, it comes as a surprise that if $n = \dim X \geq 3$, and $H \cong \mathbb{P}^{n-1}$, then $X \cong \mathbb{P}^n$. This is not accidental. The relation between a manifold and its hyperplane sections gets very tight as the dimensions increase. Indeed as n increases it becomes increasingly rare for a manifold to be a hyperplane section of another projective manifold.

To study hyperplane sections, it is natural and convenient to work more intrinsically. A line bundle, L, on a projective variety, X, is said to be *very ample* if there is an embedding $\phi: X \to \mathbb{P}^N$ for some N such that $L \cong \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ where $\mathcal{O}_{\mathbb{P}^N}(1)$ is the line bundle whose Chern class is Poincaré dual to a linear $\mathbb{P}^{N-1}$. Zero sets, H, with their multiplicities, of not identically zero sections