12 T. Yoshizawa, *Stability theory by Liapunov's second method*, Publications of the Mathematical Society of Japan, no. 9, Tokyo, 1966.

KENNETH L. COOKE
POMONA COLLEGE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, January 1992

0273-0979/92 $1.00 + $.25 per page

Classification theories of polarized varieties, by Takao Fujita. Cambridge University Press, 1990, 205 pp., $64.50. ISBN 0-521-39202-0

For simplicity we work over the complex numbers, $\mathbb{C}$, in this review. Let X be a projective variety, i.e. a reduced, algebraic subset of some projection space $\mathbb{P}^N$. Set theoretically X is defined by homogeneous polynomials. Assume moreover that X is irreducible and n-dimensional.

A natural way to try to understand X is to study a *hyperplane section*, H, of X, i.e. the intersection, $H = X \cap \mathbb{P}^{N-1}$, of X with a linear hyperplane, $\mathbb{P}^{N-1} \subset \mathbb{P}^N$, of $\mathbb{P}^N$. The hope here is that the hyperplane section is simpler than X and still contains usable information about X. This notion is quite simple—after a linear change of coordinates it is nothing more than setting one of the variables of the defining polynomials equal to 0.

Let me give some examples. Assume that the intersection is transverse and H is smooth. One of the simplest possible manifolds, H, is $\mathbb{P}^{n-1}$. If $n = \dim X = 1$, then H would be a single point and it is not too difficult to show that X is isomorphic to $\mathbb{P}^1$, $X \cong \mathbb{P}^1$. For $n = 2$, it is easy to see there are quite a few examples. One is $X \cong \mathbb{P}^2$ and H a linear $\mathbb{P}^1$ on $\mathbb{P}^2$. A second is $X \cong \mathbb{P}^2$ and H equal to a smooth conic, i.e. H is the zero set of an irreducible homogeneous polynomial of degree 2 on $\mathbb{P}^2$. A third example is X, a smooth hypersurface of degree 2 in $\mathbb{P}^3$ and H the intersection with a linear $\mathbb{P}^2 \subset \mathbb{P}^3$ that is transverse to X. Note that in this case $X = \mathbb{P}^1 \times \mathbb{P}^1$ and H can be taken to be the diagonal. Though there are many others, the very beautiful fact is that for all other examples, (X, H), with X a smooth surface, X is a Hirzebruch surface, F_r, i.e. X is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, and H is equal to a section. The Hirzebruch surfaces and their hyperplane sections are very well understood, (see [Ha, Chapter V, §2]). There is one for each integer $r \geq 0$ with $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and r the smallest integer such that there exists a section E of F_r over $\mathbb{P}^1$ with $E^2 = -r$. After this escalation of complication for $n = 2$, it comes as a surprise that if $n = \dim X \geq 3$, and $H \cong \mathbb{P}^{n-1}$, then $X \cong \mathbb{P}^n$. This is not accidental. The relation between a manifold and its hyperplane sections gets very tight as the dimensions increase. Indeed as n increases it becomes increasingly rare for a manifold to be a hyperplane section of another projective manifold.

To study hyperplane sections, it is natural and convenient to work more intrinsically. A line bundle, L, on a projective variety, X, is said to be *very ample* if there is an embedding $\phi: X \to \mathbb{P}^N$ for some N such that $L \cong \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ where $\mathcal{O}_{\mathbb{P}^N}(1)$ is the line bundle whose Chern class is Poincaré dual to a linear $\mathbb{P}^{N-1}$. Zero sets, H, with their multiplicities, of not identically zero sections of L, are called *very ample divisors.* Note that every hyperplane section is a very ample divisor and every very ample divisor is a hyperplane section for the embedding of X into projective space given by using all the sections of $\Gamma(L)$. A line bundle, L, is called *ample* is L^k is very ample for some $k > 0$. Not surprisingly, zero sets, A, with their multiplicities, of not identically zero sections of such an L, are called *ample divisors.* Thus every very ample divisor is ample, but not conversely. The simplest example is L equal to the line bundle associated to a single point on X when $n = \dim X = 1$. In this case L is always ample, but very ample as we noted above only if $X = \mathbb{P}^1$. If L has a zero set $\mathbb{P}^{n-1}$ for $n \geq 2$, and X is a manifold, then very pleasantly L is very ample and the classification of the last paragraph applies.

From the above definitions is is clear that the essence of being a projective variety is that there is a very ample line bundle. A *polarized variety* is a pair, (X, L), consisting of an ample line bundle, L, on a projective variety, X. Even if you are only interested in very ample line bundles on projective manifolds, very soon you are forced to consider singular varieties and ample bundles.

The smooth, projective, surfaces, S, with a very ample divisor, $C \cong \mathbb{P}^1$, have a striking, minimality property. To explain it, note that if X is an irreducible, n-dimensional variety, embedded in $\mathbb{P}^N$, and no hyperplane of $\mathbb{P}^N$ contains X, then

$$\deg(X) \geq N - \dim X + 1 .$$

Here $\deg(X)$ is that positive integer such that in homology X is $\deg(X)$ times a generator of $H_{2n}(\mathbb{P}^N, \mathbb{Z})$. It turns out that if S is embedded by all the sections of the line bundle, one of whose sections vanish on C, then $\deg(S) = N - \dim S + 1 = N - 1$. The pairs (X, L) with L very ample on X and with this minimum taken on for X embedded by $\Gamma(L)$ have been known since the 19th century. This led Fujita to introduce the very natural invariant $\Delta(X, L) = \deg(X) - \dim \Gamma(L) + n$, for polarized pairs, (X, L). This invariant, the *Δ-genus,* is by the simple relation noted above ≥ 0 whenever L is very ample. Fujita showed the basic fact that for polarized pairs

$$\Delta(X, L) \geq \dim\{x \in X | \text{all sections of } L \text{ vanish at } x\} + 1 ,$$

where the empty set is assumed to have dimension -1. Thus, the Δ-genus is always ≥ 0, and when it is equal to 0, L is spanned. Moreover Fujita showed that if $\Delta(X, L) = 0$, then L is very ample, and (X, L) has a simple classical structure (X is a so-called generalized cone). The first part of Fujita's book, Chapter I, is a careful account of what is known about polarized pairs in terms of the Δ-genus, and in particular there is a summary of what is known about polarized pairs with "small" Δ-genus. Also included in this is the important classification of Del Pezzo manifolds, which is due in large part to Fujita (a Del Pezzo manifold, X, is a smooth projective manifold such that $K_X = -(\dim X - 1)L$ for some ample line bundle, L).

The theory presented in Chapter 1 taken as a whole, and many of the individual results in particular, are useful and beautiful.

Chapters II and III are about adjunction theory. This is an area where there is currently a lot of activity. Large parts of the theory hold with X somewhat singular, where the singularities are allowed to be worst in the case when L is very ample. Rather than get involved with these technical details, I for simplicity, sketch the theory for L an ample line bundle on an n-dimensional, projective manifold, X. Then it turns out that either (X, L) is on a short list of "degenerate varieties," or there is a birational morphism $r: X \to X'$, such that

(a) r expresses X as a projective manifold, X', with a finite set $F \subset X'$ blown up;
(b) there is an ample line bundle, L' on X', such that $L \cong r^*L' - r^{-1}(F)$, i.e. the positive dimensional fibers, P, of r are isomorphism to $\mathbb{P}^{n-1}$ with $L_P \cong \mathscr{O}_{\mathbb{P}^{n-1}}(1)$;
(c) $K_X + (n-1)L \cong K_{X'} + (n-1)L'$ with $K_{X'} + (n-1)L'$ ample.

Because of the length of the lists of "degenerate varieties," I would not list them here and in what follows, beyond saying they are very special and relatively well understood, e.g. quadric fibrations, projective space bundles, quadrics, Del Pezzo manifolds. The point of the above is that except for a list of well understood pairs, (X, L), you can replace (X, L), with a simpler pair, (X', L') such that $K_{X'} + (n-1)L'$ is ample. If there are $n-1$ zero sets of sections of L' meeting transversely in a smooth curve, then $K_C \cong (K_{X'} + (n-1)L')_C$, and this is the only bundle with this property for such C if L is very ample and $K_{X'} + (n-1)L'$ is ample. (X', L') is called the *reduction*, or *first reduction*.

Assuming $n \geq 3$, it follows that except for a further list of "degenerate varieties," there is a birational map $\rho: X' \to Y$ such that $K_{X'}+(n-2)L' \cong \rho^*\mathscr{K}$ for some ample line bundle $\mathscr{K}$. To see how this result can be used to do projective classification of varieties let X be a submanifold of $\mathbb{P}^N$ and let g denote the genus of a smooth curve, C obtained as the transversal intersection of X and a linear $N-n+1$ dimensional subspace, $\mathbb{P}^{N-n+1} \subset \mathbb{P}^N$. If $K_{X'} + (n-2)L' \cong \rho^*\mathscr{K}$ for some ample line bundle $\mathscr{K}$, then $\deg(X) < 2g-2$. Thus the classification of those pairs with $\deg(X) \geq 2g-2$, reduces to studying special pairs on the list of "degenerate varieties." Very recently, [BFS, BS], have succeeded in carrying the adjunction process one major step forward. It is shown that, assuming $n \geq 6$, except for a further list it can be assumed that some positive power of $K_Y + (n-3)\mathscr{K}$ is spanned by global sections and gives a birational morphism. This reduces the problem, for $n \geq 6$, of classifying pairs (X, L) with $\deg(X) \geq g-1$, to the study of a list of "degenerate" pairs.

The last chapter of the book discusses some important conjectures and work in progress by a number of mathematicians. For example there is the very important conjecture of Fujita that given an n-dimensional, polarized variety (X, L), it follows that $(K_X+(n-1)L)\cdot L^{n-1} \geq -2$. Note that there is an integer, g, called the *sectional genus* of (X, L), that satisfies $2g-2 = (K_X+(n-1)L)\cdot L^{n-1}$. If there exist $n-1$ zero sets of sections of L meeting transversely in a smooth curve, C, then the genus of C is g. Fujita has shown this conjecture in a number of cases including polarized varieties (X, L), with $\dim X \leq 3$. Another topic of great interest is "how much of adjunction theory holds for ample vector bundles," For example the simplest result in this direction which follows very easily from Mori theory is that given an ample vector bundle E of rank $r = n+2$ on a projective manifold, then $X_K + \det E$ is ample. Work of Fujita and Ye and Zhang is discussed. Let me call attention to [ABW and Z] for further work done on this topic since Fujita's book appeared. Finally the book ends with some recent work on computer aided generation of polarized surfaces of low sectional genus. Here the interesting questions are not about individual examples, but about the asymptotic behavior of the set of examples as one or another of the invariants (such as the genus) of the polarized surface goes to infinity.

There is also a Chapter 0, which gathers in one place many results scattered throughout the literature. I have only given a brief overview of the many interesting results in this readable, very useful, book. This book belongs in the library of very geometer.

Acknowledgments

I would like to thank the National Science Foundation (NSF Grant DMS 89-21702) for its support.

References

[ABW] M. Andreatta, E. Ballico, and J. Wiśniewski, *On contractions of smooth algebraic varieties*, preprint, 1991.

[BFS] M. C. Beltrametti, M. L. Fania, and A. J. Sommese, *On the adjunction theoretic classification of projective varieites*, Math. Ann. **190** (1991), 31–62.

[BS] M. C. Beltrametti and A. J. Sommese, *On the adjunction theoretic classification of polarized varieties*, preprint, 1991.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., no. 52, Springer-Verlag, New York, 1977.

[Z.] Q. Zhang, *A theorem of the adjoint system for vector bundles*, Manuscripta Mathematica **70** (1991), 189–201.

Andrew J. Sommese
University of Notre Dame

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, January 1992

0273-0979/92 $1.00 + $.25 per page

The geometry and physics of knots, by Michael Atiyah. Cambridge Univ. Press, 1990, 75 pp., $39.50. ISBN 0-521-39521-6

If a_k denotes the number of k-dimensional faces of a finite polyhedron P, then $\chi(P) = \sum(-1)^k a_k$ is a topological invariant of P. This beautiful property of $\chi(P)$ goes back to Euler—or even Descartes! (see [6]), and in the 200 years since his death a large edifice of algebraic topology has grown up in which this Euler characteristic and its generalization have played a vital, and for the most part, simplifying role. Thus in the framework of cohomology $\chi(P)$ is reinterpreted as $\sum(-1)^k \dim H^k(P)$, so that the original Euler formula sheds light on some aspects of these more sophisticated vector-space valued invariants $H^k(P)$.

But it is in the domain of smooth oriented compact manifolds that this Euler number admits its most geometric interpretation. Namely, if $\Delta\colon M \to M \times M$ denotes the diagonal inclusion of M, then

$$\chi(M) = \text{self-intersection of } \Delta(M) \text{ in } M \times M.$$