examples, but about the asymptotic behavior of the set of examples as one or another of the invariants (such as the genus) of the polarized surface goes to infinity.

There is also a Chapter 0, which gathers in one place many results scattered throughout the literature. I have only given a brief overview of the many interesting results in this readable, very useful, book. This book belongs in the library of very geometer.

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References

[ABW] M. Andreatta, E. Ballico, and J. Wiśniewski, *On contractions of smooth algebraic varieties*, preprint, 1991.

[BFS] M. C. Beltrametti, M. L. Fania, and A. J. Sommese, *On the adjunction theoretic classification of projective varieites*, Math. Ann. **190** (1991), 31–62.

[BS] M. C. Beltrametti and A. J. Sommese, *On the adjunction theoretic classification of polarized varieties*, preprint, 1991.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., no. 52, Springer-Verlag, New York, 1977.

[Z.] Q. Zhang, *A theorem of the adjoint system for vector bundles*, Manuscripta Mathematica **70** (1991), 189–201.

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The geometry and physics of knots, by Michael Atiyah. Cambridge Univ. Press, 1990, 75 pp., $39.50. ISBN 0-521-39521-6

If a_k denotes the number of k-dimensional faces of a finite polyhedron P, then $\chi(P) = \sum(-1)^k a_k$ is a topological invariant of P. This beautiful property of $\chi(P)$ goes back to Euler—or even Descartes! (see [6]), and in the 200 years since his death a large edifice of algebraic topology has grown up in which this Euler characteristic and its generalization have played a vital, and for the most part, simplifying role. Thus in the framework of cohomology $\chi(P)$ is reinterpreted as $\sum(-1)^k \dim H^k(P)$, so that the original Euler formula sheds light on some aspects of these more sophisticated vector-space valued invariants $H^k(P)$.

But it is in the domain of smooth oriented compact manifolds that this Euler number admits its most geometric interpretation. Namely, if $\Delta\colon M \to M \times M$ denotes the diagonal inclusion of M, then

$$\chi(M) = \text{self-intersection of } \Delta(M) \text{ in } M \times M .$$

That is, if we place the diagonal in a generic general position relative to itself by a small perturbation, then $\chi(M)$ measures the number of times these two versions of M in $M \times M$ will intersect, intersection counted in the algebraic manner. In this domain, $\chi(M)$ therefore appears as a manifestation of "Poincaré duality" on manifolds and in the middle of this century we have slowly learned how to augment these Poincaré duality constructions. In particular, for *differentiable* manifolds higher obstructions to disentangling $\Delta(M)$ from itself in $M \times M$ lead to the Pontrjagin classes $p_k \in H^{4k}(M)$ and via their intersections

$$p^{\alpha}(M) = \int_M p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

to new numerical invariants of M.

Presumably these numbers also admit a completely "combinatorial" algorithm, and although some very interesting work has been done in this direction, see [4], a completely satisfactory account of such algorithms is still missing. Of course in view of the very satisfactory conceptual origins of the $p^{\alpha}(M)$ such a combinatorial algorithm would be more in the line of closing a chapter than opening up of a new area of inquiry.

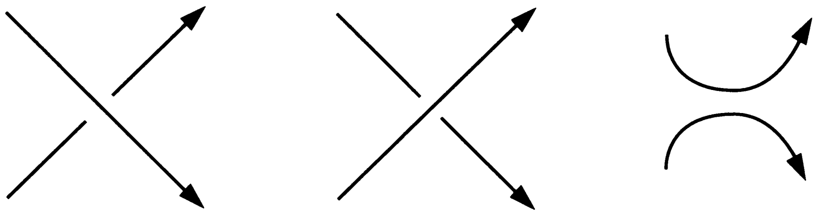
The exciting developments of the past 10 years or so is, therefore, that we are now in possession of new numerical invariants, which seem to transcend our standard algebraic topology know-how and which might well be the starting point of a more profound understanding of the diffeomorphism category.

This development started with two unexpected discoveries of the 1980s, the Donaldson four-manifold invariants, and the Vaughan Jones polynomials for knots, and combined with the exciting exchange of ideas between physics and mathematics, one is now on the threshold of overlooking a subject which is a complex hybrid of statistical mechanics, algebraic topology and the representation theory of Lie groups. It is into some aspects of these mysteries that Michael Atiyah's short book is an excellent introduction. It traces a fascinating path between these concepts with beautiful insights in many directions along the way. The exposition is taut and the writing simple, elegant, and to the point. It is in fact a tour de force of single-minded and jargon-free exposition.

Still, and this is of course the other side of the coin, to a mathematician, this little volume has a certain chimerical quality. After all, we mathematicians carry our heavy burden of ifs and buts and caveats for a good purpose: namely, to produce statements that are precise and true. We therefore like to distill our insight into theorems and then stick our neck out by asserting their eternal verity. There are no theorems in *The geometry and physics of knots*. Rather, in the prevailing manner of physics, Atiyah has here laid out the main lines and plausibility of an argument that at this stage has not been brought to a complete resolution. In short, we are dealing here more with poetry and inspirational writing than with the prose of everyday mathematics, and in this spirit, it is a pleasure to recommend this little volume to one and all.

The quest that is chronicled here is for a "conceptual" understanding of the Vaughan Jones polynomials, rather in the way in which the already-mentioned self-intersection of ΔM in $M \times M$ brings us to a more conceptual understanding of the Euler number.

An even better illustration of what we are after is to be found in the history of the Alexander polynomial of a knot. In 1928 Alexander described a combinatorial procedure of associating to a 2-dimensional knot-projection, a polynomial $A(x)$ whose nonzero roots depended only on the knot and not on the particular projection. Put differently, the projection of a knot $\underline{k}$ on the plane determined $A_{\underline{k}}(x)$ up to the equivalence $A_{\underline{k}}(x) \simeq \pm x^s A_{\underline{k}}(x)$. In this paper he also showed that $A_{\underline{k}}(x)$ satisfied a "skein-relation." That is, given a link diagram and a crossing-point, if one alters the crossing to create three different diagrams



L_+ L_- L_0

then the Alexander invariants of the resulting links are related. Because L_0 is clearly simpler than L_+ and L_- this skein relation can therefore serve to define A_L inductively for any link L.

In the 1960s Conway found a way of normalizing the Alexander polynomial to produce the Conway polynomial ∇_L, which is uniquely determined by the following simple axioms:

(1) $\nabla_{L_1 \coprod L_2} = \nabla_{L_1} \cdot \nabla_{L_2}$,
(2) $\nabla_0 = 1$,
(3) $\nabla_{L_+}(t) - \nabla_{L_-}(t) = z\nabla_{L_0}$, $z = t - t^{-1}$,

and in terms of ∇_L a representative of the classical Alexander polynomial is given by the substitution:

$$A_L(z^2) = \nabla_L(z), \qquad z = (t - t^{-1}).$$

After the fact, these combinatorial algorithms are therefore quite simple and their invariance under the "Reidemeister moves"—which are known to mediate between projections of equivalent knots—are also not too hard to check. On the other hand, the induction involved in their computation is formidable and seems to be of exponential growth in the number of crossings. They also seem altogether mysterious from the purely algorithmic point of view.

On the other hand, we now also have quite different conceptual insights into the Alexander polynomial. The first—already going back to Alexander's paper—interprets $A_{\underline{k}}(x)$ as the determinant of a linking matrix associated to the knot, and this interpretation also exhibits $A_{\underline{k}}(x)$ as the annihilator of a torsion-module over $\mathbf{Z}[T, T^{-1}]$ associated to the knot $\underline{k}$ by the following geometric construction. Consider $\pi_1 = \pi_1(S^3 - \underline{k})$, the fundamental group of the knot complement, and let $M(\underline{k})$ be the covering of $S^3 - \underline{k}$ associated to the commutator subgroup $[\pi_1, \pi_1]$ in π_1.

The group $\pi_1/[\pi_1, \pi_1]$ then acts on $M(\underline{k})$ via the "deck-transformations," so that $H^1(M(\underline{k}); \mathbf{Z})$ is naturally a module over the group ring of $\pi_1/[\pi_1, \pi_1]$, which—if we assume $\underline{k}$ connected—reduces to $\mathbf{Z}[T, T^{-1}]$. The annihilator of the module $H^1(M(\underline{k}))$ qua $\mathbf{Z}(T, T^{-1})$ now determines an element $\tilde{A}_{\underline{K}}(T) \in \mathbf{Z}[T, T^{-1}]$, which, up to the ambiguity already discussed, is seen to represent $A_k(T)$! This formulation therefore fits the Alexander polynomial squarely into the mainstream of algebraic topology.

Now the Jones polynomial $V_{\underline{k}}(t)$ of a knot also satisfies a deceptively simple skein relation, to wit:

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}$$

but so far the *new* polynomial has defied an algebraic-topology formulation of the type we just found for its older cousin. Its conceptual roots seem to lie in a discipline far removed from algebraic topology—namely, in statistical mechanics! and that is most probably the reason why its discovery took so long. Vaughan Jones of course at first came upon $V_L(t)$ unexpectedly from quite a different perspective—von Neumann's algebras and braids—but soon was able to interpret $V_L(t)$ in terms of statistical mechanics, and indeed, thereafter Kaufmann also produced a state-model for the Alexander polynomial [5].

Let me recall here that in a typical statistical mechanics model, one starts with a configuration space X (think of a finite square lattice) and a set S of internal states—or colours—for the elements of X (think of spin, where $S = \pm 1$). A colouring of X—or state of the system—then amounts to the assignment of a "spin" $s(x)$ to each site of $x \in X$.

The colourings are therefore elements of the function-space $\mathrm{Map}(X; S)$. Now the "energy" of a colouring (s) is then given by an algorithm for computing a number $E(s)$ from the deposition of s, and in the simplest models, $E(s)$ will depend only on the nearest neighbor terms. This $E(s)$ will also depend on an auxiliary parameter, say T, which in practice is related to temperature.

These data therefore yield a "partition function"

$$Z(T) = \sum_{s \in \mathrm{Map}(X, S)} E_T(s).$$

Hence, to write down a state-model for the Alexander or Jones polynomials consists of devising a procedure which converts a link diagram to a statistical ensemble whose partition function yields the polynomial in question. I cannot resist a personal diversion here. In a very early paper [2]—in fact, just after learning and meditating on the Euler number in 1949—I devised a new combinatorial invariant for polyhedra, which, I realize now in retrospect, was derived precisely by such a "state model" construction. I am afraid this invariant has rested very peacefully in the literature these past 40 years, but who knows, maybe its day is at hand. It seemed to me at that time only a curiousity with no connections to anything and not interesting for manifolds; for it is mainly concerned with the ways in which cells of the top dimensional hang together.

But to return to the subject at hand. The statistical model I have just described carries with it—so to speak, trivially—a certain functorial point of view which it took Witten and Graeme Segal to truly distill and explain to mathematicians at large, and which we now call "Topological Quantum Field Theory." The starting point of this development is the following observation concerning any finite statistical model. Let Y_0 and Y_1 be two disjoint subsets of X. Then given a colouring s_0 of Y_0 and a colouring s_1 of Y_1 we can consider the set $\pi^{-1}(s_0, s_1)$ of colourings $\{s\}$ of Y which *extend* s_0 and s_1, and sum their energies to obtain a number which we notate

$$\langle s_0, s_1 \rangle = \int_{s|Y_0 = s_0,\, s|Y_1 = s_1} E(s)\mathscr{D}(s).$$

Now let us write $V(Y_0)$ and $V(Y_1)$ for the free vector space spanned by the set of respective colourings of Y_0 and Y_1. Then our integral clearly extends to a bilinear function

$$V(Y_0) \otimes V(Y_1) \to \mathbf{R},$$

or, quite equivalently to a linear map

$$V(Y_0) \to V^*(Y_1),$$

which in the physics world is often called a "propagator." Note also that in the language of measure theory this construction simply amounts to *the push-forward nature of a measure.* Indeed, if $V(X)$ denotes the space of real valued functions on $\operatorname{Map}(X, S)$ then $E(s)\mathscr{D}(s)$ can be thought of as defining a measure on $\operatorname{Map}(X, S)$ with the total measure of $\operatorname{Map}(X, S)$ given by

$$Z(X) = \int_{\operatorname{Map}(X, S)} 1E(s)\mathscr{D}(s),$$

that is, precisely the partition function of X.

The natural projection π of $\operatorname{Map}(X; S)$ to $\operatorname{Map}(Y_0 \coprod Y_1; S)$ given by restriction, thus pushes $E(s)\mathscr{D}(s)$ to a measure on $\operatorname{Map}(Y_0 \coprod Y_1; S) \equiv \operatorname{Map}(Y_0, S) \times \operatorname{Map}(Y_1, S)$ and so extends to the map of $V(Y_0) \otimes V(Y_1) \to \mathbf{R}$ indicated already. Note that the mapping space construction here intervenes to convert *additive operations to multiplicative* ones so that topological axioms built on this model naturally seem to be *multiplicative* analogues of the usual ones.

In the topoligical context X is usually taken to be an n-manifold of which Y_0 and Y_1 are typically two components of the boundary, and a complete axiom system linking the number $Z(X)$ with the vector spaces $V(Y_0)$ and $V(Y_1)$ is discussed in the second chapter of this book. The reader might find it profitable to check those axioms in the baby context of the present discussion.

The true aim of statistical mechanics, and to a large extent of quantum field theory, is to deal with the behaviour of the lattice model as the lattice tends to infinity. For instance, phase transitions are characterized by the fact that $Z(T)$ which is usually quite analytic in T for finite X, develops singularities with the passage to ∞.

In field theories the physicists often write down a "measure" on an infinite-dimensional space—purely by analogy and "physical intuition." Thus the statistical model which plays a central role in this volume is the Witten-Wes-Zumino model whose partition function has the following form:

$$Z_k^G(M) = \int_{\mathscr{A}} e^{2\pi i k L(A)} \mathscr{D}A$$

with

$$L(A) = \frac{1}{8\pi^2} \int_M \operatorname{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A).$$

Here M is a compact, 3-manifold and plays the role of the configuration space. A "colouring" of M, for $P = M \times G$ is a choice of a fixed "connection" A on the principal bundle P over M with structure group G. A is therefore simply a Lie algebra valued 1-form on M. Thus $\operatorname{Map}(X, S)$ is here interpreted by the set "$\mathscr{A}$" of all such connections. The action of the colouring A is now measured by the "Chern-Simons" Lagrangian $L(A)$. Note here that because A is a 1-form, $(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ is naturally a Lie algebra valued 3-form so that the integral is well defined on the compact 3-manifold. In the language of physics $L(A)$ is the Lagrangian of the theory and the corresponding integral $Z(M)$,

$$Z(M) = \int_{\mathscr{A}} e^{2\pi i k L(A)} \mathscr{D}(A)$$

is meant as a modern version of the Feynmann integral "over all paths". The "temperature" of this theory is given by the parameter k.

Actually the form of this integral is a beautiful nonabelian generalization of a formula going back to Dirac in the abelian case where G is the circle S^1, and corresponds to the electromagnetic force. The difficulty of defining the infinite dimensional integral $\int_{\mathscr{A}}$ is compounded in these "gauge theories" by the fact that properly speaking "colourings" A and A^g are to be considered equivalent if they are related by a gauge transformation g, that is, if we can construct a map $g\colon M \to G$, such that

$$A^g \equiv gAg^{-1} + g^{-1}\,dg\,.$$

The expression $L(A)$ is cunningly constructed so as to be "nearly" invariant under such gauge transformations:

Indeed, if $\mathscr{G} = \mathrm{Map}(M\,;\,G)$ denotes this *gauge group*, then

$$L(A^g) = L(A)$$

for g in the *identity component* of $\mathscr{G}_0$ of $\mathscr{G}$ and in general $L(A^g) - L(A)$ is an integer counting the degree of $g\colon M \to G$ relative to the 3-dimensional generator of $H^3(G)$.

Hence it is only if k is an *integer*, that $e^{2\pi i k L(A)}$ can be thought of as a function on $\mathscr{A}/\mathscr{G}$, and it is only then that these formulae are expected to make "real sense." This "quantization" of k is precisely Dirac's argument for the quantization of electric charge: The infinite integral *must* make sense by *Physics*, hence k is an integer. Q.E.D!

In short, if we invoke *Physics*, at this stage of the game, the W-Z-W statistical model should produce for every $k \in \mathbf{Z}$ and compact group G, a numerical invariant $Z_k^G(M)$, of the 3-manifold M. And, indeed, in [7] Witten "computes" this invariant for the 3-sphere S^3 and the group $G = SU(2)$ to be

$$Z_k^G(S^3) = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{\pi}{k+2}\right).$$

Even granting this formula the reader might now well ask what this result has to do with knots, and the answer Witten produces is exceedingly appealing. Namely, in statistical mechanics it is not only the partition function that counts. Rather, the measure on $\mathrm{Map}(X\,,\,S)$ provides one with *expectation values* $\langle f\rangle$ for functions over f on $\mathrm{Map}(X\,,\,S)$. That is, we set

$$\langle f\rangle = \int_{\mathscr{A}/\mathscr{G}} f(s)\mathscr{D}(s)/Z(X)\,.$$

And in this spirit the knot enters the picture in Witten's model via the functions it naturally defines on our space $\mathscr{A}/\mathscr{G}$.

Recall that $\mathscr{A}$ was a space of connections on P over M. Given A, any closed oriented curve $\underline{k}$ in M therefore gives rise to a holonomy element $(g_{\underline{k}})$ in G which measure the effect of parallel transport "according to A" along $\underline{k}$.

Actually this element is well defined only modulo an inner automorphism of G, so that to obtain a well-defined function on $\mathscr{A}/\mathscr{G}$, Witten chooses an auxiliary representation V of G, and defines

$$\chi_W(\underline{k}) = \text{Trace}\, \rho(g_{\underline{k}})$$

as the " W-measure" of the holonomy determined by the oriented knot $\underline{k}$. In this way every pair (G, W) defines a numerical invariant on knots by its expectation value $\langle \chi_W(\underline{k}) \rangle$ in the k-G theory, and Witten argues that these expectation values are related to the values of the Jones and Alexander polynomials at various roots of unity.

For instance, when $G = SU(2)$ and W is the representation of "level k," then according to Witten

$$\langle \chi_W(\underline{k}) \rangle = V_{\underline{k}}(q), \qquad q = e^{2\pi i/2+k},$$

and I hope that the urge of discovering what is behind this intriguing explicit formula will propel the reader unhesitatingly to the bookstore for a copy of Atiyah's, *The geometry and physics of knots.*

References

1. J. W. Alexander, *Topoligical invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275.
2. R. Bott, *Two new combinatorial invariants for polyhedra*, Portugal. Math. **2** (1952), pp. 35–40.
3. S. Donaldson, *An application of gauge theory to the topology of four manifolds*, J. Differential Geom. **18** (1983), 269; *Polynomial invariants for smooth four-manifolds*, Oxford preprint.
4. R. MacPherson, *The combinatorial formula of Gabrielov, Gelfand, and Losik for the first Pontrjagin class*, Séminaire Bourbaki No. 497, Lecture Notes in Math., vol. 667, Berlin and New York, 1977.
5. L. Kaufman, *State models and the Jones polynomial*, Topology, **26** (1987), 395–407.
6. I. Lakatos, *Proofs and refutations*, Cambridge Univ. Press, London and New York, 1976.
7. V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 103–111.
8. E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

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Variational principles of topology. Multidimensional minimal surface theory, by Anatoliĭ T. Fomenko. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1990, 374 pp., $133.00. ISBN 0-7923-0230-3

Anatoliĭ T. Fomenko is the most prominent mathematician in the Soviet Union working in higher dimensional minimal surface theory. The book under review on area minimizing surfaces is a revised translation of his earlier volume