APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, Jan 1992, Pages 188-192

BOOK REVIEW

Variational principles of topology. Multidimensional minimal surface theory, by Anatoliĭ T. Fomenko. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1990, 374 pp., $133.00. ISBN 0-7923-0230-3

Anatoliĭ T. Fomenko is the most prominent mathematician in the Soviet Union working in higher dimensional minimal surface theory. The book under review on area minimizing surfaces is a revised translation of his earlier volume *Variational methods in topology* (published in Russian in 1982) based on a course which he taught to undergraduate and graduate students at Moscow State University. In 1990 Fomenko published a two volume work, *Plateau's problem* [F1], as a general survey of problems of least area in contrast with the work being reviewed which focuses mainly on his own contributions. Fomenko is also an accomplished artist; some of his work was recently published by the Society [F2].

The study of least area problems in general dimensions is largely a development of twentieth century mathematics. Prior to about 1960 a k-dimensional area minimizing surface in an n-dimensional Riemannian manifold likely would have been either a submanifold or the image of a k-dimensional manifold under a reasonable mapping. In the first case surface area would have been Hausdorff's intrinsic k-dimensional measure (the usual surface area of a submanifold) while in the latter case, area would have been the integral of the Jacobian of the mapping. Except when $k = 2$ there were essentially no existence theorems for area minimizing surfaces, at least with any regularity.

The year 1960 is frequently taken as a turning point in the geometric calculus of variations, and especially in the study of least area, because of three seminal contributions.

First was the paper *Normal and integral currents* [FF] by H. Federer and W. H. Fleming; this paper was awarded one of the Society's 1987 Steele Prizes. The best introductory reference to the mathematics which grew from this start is the book *Geometric measure theory. A beginner's guide* [MF] by F. Morgan.

A second contribution was work of E. De Giorgi which included, in particular, an almost everywhere regularity theorem for area minimizing oriented hypersurfaces. Even though this theory is formally a mathematical subset of geometric measure theory, it has its own structure and, for the study of area minimizing hypersurfaces, is more readily accessible. The best introductory reference is the book *Minimal surfaces and functions of bounded variation* [GE] By E. Giusti (incorrectly spelled Ciusti in the volume under review).

The third contribution in 1960 was the paper *Solution of the Plateau Problem for m-dimensional surfaces of varying topological type* [R1] by E. R. Reifenberg which gave the first almost everywhere manifold structure for area minimizing surfaces of general codimension. Reifenberg, who was killed in a mountaineering accident in the Dolomites in the mid 1960s, was a student of A. S. Besicovitch in whose pioneering works lie many roots of present theories. Fomenko's work is largely set in an extension of Reifenberg's context.

In each of these approaches and contexts, area minimizing surfaces are sought among surfaces of varying topological type and singularity structure. Some such freedom is necessary in any general theory because not every boundary or homology class can be spanned by a singularity free minimal surface, as cobordism theory shows [AB]. There are, however, significant differences between these approaches. Within the geometric measure theory school, surfaces are often "integral currents," de Rham currents which are strongly approximatable by Lipschitz singular chains. These currents form a chain complex with the same homology groups as singular chains and have strong approximation and compactness properties; these properties guarantee the existence of surfaces representing integral homology classes and minimizing integrals of elliptic parametric integrands. Integrals, including area, being minimized are usually computed with densities equal to topological multiplicity. Area weighted with such densities is called "mass" while area without multiplicity is sometimes called "size"—both are lower semicontinuous under weak convergence of currents, although a mass bound typically is necessary in order that a minimizing sequence have a convergent subsequence. It is a theorem of the reviewer that any mass minimizing integral current is a minimal submanifold except for a possible singular set of codimension at least two [A3]; this is the best possible general result since holomorphic varieties in Kähler manifolds are mass minimizing. The singular sets of mass minimizing oriented hypersurfaces are of codimension at least seven [FH] according to earlier work of the reviewer, E. Bombieri, E. De Giorgi, H. Federer, W. H. Fleming, E. Giusti, and J. Simons.

For Reifenberg, a surface was a compact set, and its area equaled its spherical Hausdorff measure, analogous to size rather than mass. These surfaces span their boundaries or homology classes in the sense of Čech homology. Reifenberg obtained the first regularity results for area minimizing surfaces in general dimensions and codimensions [R2] in 1964 by showing in $\mathbf{R}^n$ that his surfaces were almost everywhere smooth minimal submanifolds. His surfaces typically have singularities of codimension one along which three sheets of surface meet at equal angles as in soap films. Because area is not counted with multiplicities, the branching singularities of mass minimizing integral currents (as in holomorphic varieties) do not occur in his surfaces. Since Fomenko works in Reifenberg's context, when he says a minimal surface is the image of a manifold, the area in question is the Hausdorff measure of the image and not the Jacobian mapping area mentioned above.

There are several other types of measure theoretic surfaces in substantial use today. Flat chains $\bmod \nu$ (introduced in [ZW] in 1962 and in [FW] in 1966) are modelled on Lipschitz singular chains with coefficients in the integers modulo ν; an area minimizing Möbius band would thus be a flat chain $\bmod 2$. Varifold surfaces (introduced in [A1] in 1965 and in [AW] in 1972) are useful for the variational calculus in the large and motion by mean curvature [BK]; every compact Riemannian manifold, for example, contains a minimal hypersurface whose singular set has codimension at least seven [PJ]. The $(\mathbf{F}, \varepsilon, \delta)$ minimal surfaces (introduced in [A2] in 1976) are useful for constrained surface energy minimization such as occurs in soap bubble clusters; in mathematical models for soap films and soap bubble geometries as $(\mathbf{M}, \varepsilon, \delta)$ minimal sets, typically size is minimized and singular sets consist of smooth curves meeting at equal angles at isolated points according to J. Taylor [T1]. Multifunctions (introduced in [A3] in 1984 and [A4] in 1988) are useful for analysis of branching behavior of minimal surfaces [CS] and in the study of rectifiable currents with real densities.

Fomenko's book touches on a large number of topics in minimal surface theory; the table of contents is seven pages long! He lists three principal themes:

(1) *Stratified surfaces and stratified volumes*. One of Fomenko's main original contributions to the multivariable variational calculus is his introduction of interesting new constraints for area minimization called spectral homology and cohomology. This constitutes the central topic of the book. Roughly speaking, what is required for such a constraint is preservation of nontriviality under convergence of compact sets in the Hausdorff distance topology. Since the smallest set carrying the nontriviality can be of lower dimension (the nontriviality of the normal bundle of a Möbius band in $\mathbf{R}^3$, for example, is carried by a curve) one is led to construct strata of minimal surfaces, each minimizing area in its own dimension subject to the condition that all higher-dimensional pieces already minimize their area. Fomenko proves the existence and almost everywhere regularity of each stratum by an extension of Reifenberg's arguments; regularity alternatively follows from the reviewer's paper [A2].

(2) *A new method for the proof of global minimality and new examples*. Some of Fomenko's especially nice contributions are examples of absolutely area minimizing surfaces in Lie groups and an apparent complete classification of locally minimal, totally geodesic submanifolds realizing nontrivial cycles and elements of homotopy groups in symmetric spaces.

(3) *A relation between Bott periodicity, topologically nontrivial, totally geodesic surfaces in Lie groups, and the number of independent vector fields on spheres*.

The reviewer has known Fomenko personally for more than two decades and still is at a loss to understand why he is not more responsible in his mathematical claims. The following are two particular examples of concern.

The book cover states "In this volume, the solution of the Plateau problem in the class of all manifolds with fixed boundary is given in detail … " Fomenko made a similar claim in a lecture at and in the proceedings of the 1974 International Congress in Vancouver, in the introduction to a major paper (in Russian), and in an interview published in the Mathematical Intelligencer. His preface in the volume under review is ambiguous about this issue. In any case, the claim is not proved, as he acknowledges privately. It is not known at present whether or not minimal surfaces of the type he studies are necessarily representable as continuous images of manifolds or even as continuous images of sets of finite topological complexity. The only significant contributions to this representation problem are due to B. White [W1, W2] who worked in a somewhat different mathematical context.

A second example occurs in §8 of Chapter 2, entitled *Solution of the problem of finding globally minimal surfaces in each homotopy class of multivarifolds*. Fomenko asserts "Ɖ̀ao Chông Thi solved Plateau's problem by establishing the existence of a locally Lipschitz mapping $g_0 \colon W^k \to M^n$ in terms of currents, which minimizes the k-dimensional volume functional in the class of all locally Lipschitz mappings $g \colon W \to M$ such that $g|_{\partial W} = f|_{\partial W}$ (the problem of finding the absolute minimum with respect to all homotopy classes of multivarifolds)." In fact, Thi did not prove such a theorem in papers known to the reviewer since he did not establish a common

Lipschitz constant to his sequence of mappings. Both the reviewer and others have pointed this out to Fomenko in person. Yet he again makes his claim!

At this present time, the geometric calculus of variations is thriving theoretically and computationally. An overview of the theory as of 1984 can be found in the table of contents of the volume *Geometric measure theory and the calculus of variations* [AA]. A sampling of current developments such as mean curvature evolution and the crystalline variational calculus appears in the video tape and proceedings, *Computing optimal geometries* [T2]. The reviewer's paper *Questions and answers about area minimizing surfaces and geometric measure theory* [A5] is also intended as an overview of where we are and where we would like to go.

References

[AW] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[AA] W. K. Allard and F. Almgren, eds., *Geometric measure theory and minimal surfaces*, Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986.

[A1] F. Almgren, *The theory of varifolds. A variational calculus in the large for the k dimensional area integrand*, multilithed notes (no longer available), 1965; see [AW].

[A2] ———, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. No. 165 (1976).

[A3] ———, **Q** *valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two*, preprint, 1984. See Bull. Amer. Math. Soc. (N.S.) **8** (1983), 327–328.

[A4] ———, *Deformations and multiple-valued functions*, Geometric Measure Theory and the Calculus of Variations, Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 29–130.

[A5] ———, *Questions and answers about area minimizing surfaces and geometric measure theory*, Proc. 1990 AMS Summer Research Institute on Differential Geometry.

[AB] F. Almgren and W. Browder, *On smooth approximation of integral cycles*, (in preparation).

[BK] K. A. Brakke, *The motion of a surface by its mean curvature*, Math. Notes, no. 20, Princeton Univ. Press, Princeton, NJ, 1978.

[CS] S. Chang, *Two dimensional area minimizing currents are classical minimal surfaces*, J. Amer. Math. Soc. **1** (1988), 699–778.

[FH] H. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat-chains modulo two with arbitrary codimensions*, Bull. Amer. Math. Soc. **76** (1970), 767–771.

[FF] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520.

[FW] W. H. Fleming, *Flat chains over a coefficient group*, Trans. Amer. Math. Soc. **121** (1966), 160–186.

[F1] A. T. Fomenko, *The Plateau problem.* Part I. *Historical survey.* Part II. *The present state of the theory*, Studies in the Development of Modern Mathematics, Gordon and Breach, New York, 1990.

[F2] ———, *Mathematical impressions*, Amer. Math. Soc., Providence, RI, 1990.

[GE] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs Math., vol. 80, Birkhäuser, Boston-Basel-Stuttgart, 1984.

[MF] F. Morgan, *Geometric measure theory. A beginner's guide*, Academic Press, New York, 1987.

[PJ] J. T. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Math. Notes, no. 27, Princeton Univ. Press, Princeton, NJ, 1981.

[R1] E. R. Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. **104** (1960), 1–92.

[R2] ———, *A epiperimetric inequality related to the analyticity of minimal surfaces. On the analyticity of minimal surfaces*, Ann. of Math. (2) **80** (1964), 1–21.

[T1] J. E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) **103** (1976), 489–539.

[T2] J. E. Taylor, ed., *Computing optimal geometrices*, Amer. Math. Soc., Providence, RI, 1991.

[W1] B. White, *Existence of least area mappings of N-dimensional domains*, Ann. of Math. (2) **18** (1983), 179–185.

[W2] ______, *Mappings that minimize area in their homotopy classes*, J. Differential Geom. **20** (1984), 433–446.

[ZW] W. P. Ziemer, *Integral currents mod* 2, Trans. Amer. Math. Soc. **105** (1962), 496–524.

Fred Almgren
Princeton University
The National Science and Technology Research Center
for Computation and Visualization of Geometric Sturcture