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SEMISTABILITY OF AMALGAMATED PRODUCTS, HNN-EXTENSIONS, AND ALL ONE-RELATOR GROUPS

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1. Introduction

Semistability at infinity is a geometric property used in the study of ends of finitely presented groups. If a finitely presented group G is semistable at infinity, then sophisticated invariants for G, such as the fundamental group at an end of G, can be defined (see [10]). It is unknown whether or not all finitely presented groups are semistable at infinity, although by [16] it suffices to know whether all 1-ended finitely presented groups are semistable at infinity. There are a number of results showing many 1-ended groups have this property, e.g., if G is finitely presented and contains a finitely generated, infinite, normal subgroup of infinite index, then G is semistable at infinity (see [12] and for other such results [13–15]).

Semistability at infinity is of interest in the study of cohomology of groups; if a finitely presented group G is semistable at infinity, then $H^2(G; \mathbb{Z}G)$ is free abelian (see [6, 7]). This is conjectured to be true for all finitely presented groups, but at present it is not even known for 2-dimensional duality groups (where one is discussing the dualizing module, see [2]).

For negatively curved groups (i.e., hyperbolic groups in the sense of Gromov, see [8]), semistability at infinity has additional interesting consequences. If a negatively curved group G is given the word metric with respect to some finite generating set, then there is a compactification $\overline{G}$ of G where a point of $\partial G = \overline{G} - G$ is a certain equivalence class of proper sequences of points in G. The boundary of G is a compact, metrizable, finite-dimensional space, which determines the cohomology of G. Bestvina and Mess have shown that if G is a negatively curved group, then for every ring R, there is an isomorphism of RG-modules $H^i(G; RG) \cong \check{H}^{i-1}(\partial G; R)$ (Čech reduced). Geoghegan has observed that results in [1] imply that a negatively curved group G is semistable at infinity iff ∂G has the shape of a locally connected continuum (see [6]). Furthermore, in [1], ideas closely related to semistability at infinity are used to analyze closed irreducible 3-manifolds with negatively curved fundamental group.

A continuous map is *proper* if inverse images of compact sets are compact. Proper rays r, $s\colon [0,\infty) \to K$ in a locally finite CW-complex K are said to *converge to the same end* of K if for every compact $C \subseteq K$ there exists an N such that $r([N,\infty))$ and $s([N,\infty))$ are contained in the same path component of $K - C$. A locally finite

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CW-complex K is *semistable at infinity* if any two proper rays, which converge to the same end of K, are properly homotopic. If G is a finitely presented group, then G is *semistable at infinity* if for some (equivalently any) finite CW-complex X with $\pi_1(X) = G$, the universal cover $\widetilde{X}$ of X is semistable at infinity.

If H is a subgroup of two groups A and B, the amalgamated product $A *_H B$ is the quotient of the free product of A and B where the copies of H in A and B are identified. If H and H' are isomorphic subgroups of A, the HNN-extension $A*_H$ (where H' is taken as given) is the quotient of $A * \langle t \rangle$ where H is identified with $tH't^{-1}$ (see [11]). In [21], Stallings proves a decomposition theorem for finitely generated groups having more than one end in terms of amalgamated products or HNN-extensions over finite subgroups. In [4], Dunwoody shows that for finitely presented groups, the process of recursively applying this decomposition theorem to the factor groups eventually terminates in 0-ended (i.e., finite) and 1-ended factor groups. Our main result is the following:

Theorem 1. *If $G = A *_H B$ is an amalgamated product where A and B are finitely presented and semistable at infinity, and H is finitely generated, then G is semistable at infinity. If $G = A*_H$ is an HNN-extension where A is finitely presented and semistable at infinity, and H is finitely generated, then G is semistable at infinity.*

If G is the fundamental group of a graph of groups (see [20]), then G can be expressed as some combination of amalgamated products and HNN-extensions of the vertex groups over the edge groups. Hence, if G is the fundamental group of a finite graph of groups in which each vertex group is finitely presented and semistable at infinity and each edge group is finitely generated, then G is semistable at infinity. However, it is possible that a group G can be expressed as a combination of amalgamated products and HNN-extensions of finitely presented groups over finitely generated (but not finite) groups without G being the fundamental group of a graph of groups with these vertex and edge groups, hence the above theorem applies to a larger class of group decomposition. Although the question of semistability at infinity for all finitely presented groups reduces to the same question for 1-ended groups, it is possible to obtain a 1-ended group $G = A *_H B$ where A, B, and H are infinite-ended (and similarly for HNN-extensions), and in fact this is the essential difficulty in the proof of our main theorem.

As a corollary to the proof of Theorem 1, the same methods apply (with homotopy replaced by homology in the sense of [7]) to give a cohomology version of this result.

Corollary 2. *If $G = A *_H B$ is an amalgamated product where A and B are finitely presented, $H^2(A, \mathbb{Z}A)$ and $H^2(B; \mathbb{Z}B)$ are free abelian, and H is finitely generated, then $H^2(G; \mathbb{Z}G)$ is free abelian. If $G = A*_H$ is an HNN-extension where A is finitely presented, $H^2(A; \mathbb{Z}A)$ is free abelian, and H is finitely generated, then $H^2(G; \mathbb{Z}G)$ is free abelian.*

As an application of our main result, we get the following general theorem:

Theorem 3. *All finitely generated one-relator groups are semistable at infinity.*

Finally, as a corollary (using [7] as before), we get a purely cohomological result.

Corollary 4. *If G is a finitely generated one-relator group, then $H^2(G; \mathbb{Z}G)$ is free abelian.*

FIGURE 1

2. Outline of proofs

We describe the proof of our main theorem in the amalgamated product case. Take a presentation P for $G = A *_H B$ by combining presentations for A and B, each containing generators for H. If Z is the standard 2-complex obtained from P, then $Z = X \cup Y$ where X and Y are subcomplexes of Z with $\pi_1(X) = A$ and $\pi_1(Y) = B$, and $X \cap Y$ is a wedge of circles representing generators for H in both $\pi_1(X)$ and $\pi_1(Y)$. The universal cover $\widetilde{Z}$ of Z is a union of copies of $\widetilde{X}$ and $\widetilde{Y}$ attached along copies of the Cayley graph Γ of H. The group G acts on the left of $\widetilde{Z}$, permuting copies of $\widetilde{X}$, $\widetilde{Y}$, and Γ.

To prove the main theorem, we show that any two proper edge paths r and s in $\widetilde{Z}$, converging to the same end of $\widetilde{Z}$, are properly homotopic. The normal form structure of $A *_H B$ provides the geometric structure to show that r and s are properly homotopic in case $\mathrm{im}(r) \cup \mathrm{im}(s)$ intersects no copy of Γ in an infinite set of vertices.

If r or s meets some copy of Γ, say Γ_0, in infinitely many vertices, then (by replacing each ray with a properly homotopic ray passing through these points) we may as well assume $V = \mathrm{im}(r) \cap \mathrm{im}(s) \cap \Gamma_0$ contains infinitely many vertices. Let q be a proper edge path in Γ_0 passing through infinitely many vertices in V. Then q and r (and s) converge to the same end of $\widetilde{Z}$, and it suffices to show that q and r are properly homotopic (since then q and s are similarly properly homotopic, and thus r and s are properly homotopic). Thus we are reduced to the case where one of our rays is contained in a copy Γ_0 of Γ.

The main ideas in this, the main case in our work, are as follows. We split $\widetilde{Z}$ into two connected pieces $\widetilde{Z}^+$ and $\widetilde{Z}^-$, which intersect along Γ_0 by taking $\widetilde{X}_0$ and $\widetilde{Y}_0$ to be the copies of $\widetilde{X}$ and $\widetilde{Y}$ containing Γ_0 and then defining $\widetilde{Z}^+$ to be the component of $(\widetilde{Z} - \widetilde{Y}_0) \cup \Gamma_0$ containing Γ_0, and $\widetilde{Z}^-$ to be the component of $(\widetilde{Z} - \widetilde{X}_0) \cup \Gamma_0$ containing Γ_0. By extracting ideas from the proof of Dunwoody's accessibility theorem [4], we show that a certain configuration of rays and ends cannot occur in $\widetilde{Z}^+$ or $\widetilde{Z}^-$. (This configuration is represented in Figure 1, where C is a compact set in $\widetilde{Z}$; u, v, u'_i, and v'_i are proper rays in Γ_0, with the u'_i and v'_i in different components of $\Gamma_0 - C$ and diverging from u and v at progressively later points, and where ovals represent distinct ends of either $\widetilde{Z}^+$ or $\widetilde{Z}^-$.) Because this configuration cannot occur, we can construct

proper homotopies between any proper ray in Γ_0 and any proper ray in $\widetilde{Z}^+$ or ($\widetilde{Z}^-$) that converge to the same end of $\widetilde{Z}^+$ (respectively, $\widetilde{Z}^-$). In essence, this says that the ends of $\widetilde{Z}^+$ and $\widetilde{Z}^-$, determined by Γ_0, are semistable at infinity. This fact provides the geometric structure needed to construct a patchwork of proper homotopies in $\widetilde{Z}$, giving a proper homotopy between r and q and, thus, between the given r and s.

The proof that all one-relator groups are semistable at infinity is by an induction argument patterned after the proof by Magnus of the Freiheitssatz (see [11]). The proof makes use of our main theorem, the following structure theorem for one-relator groups, and a simple fact about semistability at infinity for factor groups in certain amalgamated products.

Lemma 5. *Given any finitely generated one relator group G, there exists a finite sequence of finitely generated one relator groups $H_1, H_2, \dots, H_n = G$ such that, for each $i < n$, either H_{i+1} or $H_{i+1} * \mathbb{Z}$ is an HNN-extension of H_i over a finitely generated group, and such that H_1 is either a free group or else is isomorphic to a free product of a free group and a finite cyclic group.*

Lemma 6. *If G is finitely presented and $G * \mathbb{Z}$ is semistable at infinity, then G is semistable at infinity.*

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