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BOOK REVIEW

A. Ženišek: *Nonlinear elliptic and evolution problems and their finite element approximations*, by A. Ženišek. Academic Press, London 1990, 422 pp., \$45.00. ISBN 0-12-779560-X

For a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ consider the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} b_i(x, u, \nabla u) + b_0(x, u, \nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where b_i: $\Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $i = 0, 1, 2$, are smooth functions satisfying the ellipticity and growth conditions

$$\sum_{i,j=0}^{2} \frac{\partial b_i}{\partial \xi_j}(x, \xi)\eta_i\eta_j \geq \alpha(\eta_1^2 + \eta_2^2) \quad \forall x \in \Omega,\ \forall \xi, \eta \in \mathbb{R}^3; \tag{2}$$

$$\left|\frac{\partial b_i}{\partial x_j}(x, \xi)\right| + |b_i(x, \xi)| \leq c(1 + |\xi|), \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^3; \tag{3}$$

$$\left|\frac{\partial b_i}{\partial \xi_j}(x, \xi)\right| \leq c \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^3. \tag{4}$$

Examples of functions satisfying (2)–(4) are linear elliptic operators such as the Laplacian ($b_0 = 0, b_i = \partial_i u, i = 1, 2$) or the more general operator

$$-\sum_{i,j=1}^{2} \partial_i(k_{ij}(x)\partial_j u) \tag{5}$$

with a positive definite matrix $(k_{ij}(x))_{i,j=1,2}$ for all $x \in \Omega$. Due to the restrictive growth conditions (3), (4) there are not many examples of nonlinear problems. A typical equation is the stationary magnetic field, where $b_i = \nu(x, |\nabla u|)\partial_i u$ and ν is the permeability.

In order to approximate problem (1) by the finite element method we introduce the weak formulation that is obtained by multiplying (1) with a function v with $v|_{\partial\Omega} = 0$ and using integration by parts.

Find $u \in H_0^{1,2}(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^{1,2}(\Omega), \tag{6}$$

where

$$a(u,v) = \int_\Omega \left\{ \sum_{i=1}^{2} b_i(x,u,\nabla u)\partial_i v + b_0(x,u,\nabla u)v \right\} dx$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. The space $H_0^{1,2}(\Omega)$ consists of all functions with generalized first derivatives in $L^2(\Omega)$ that vanish on $\partial\Omega$. It can easily be verified that problems (1) and (6) are equivalent if the corresponding solutions are sufficiently smooth.

The finite element method for approximating (1) consists in restricting the weak formulation to a finite-dimensional space of continuous and piecewise polynomial functions. At first, we describe the space of piecewise linear shape functions. Let Π_h be a triangulation of the polygonal domain Ω into triangles with exterior diameter $c_1 h$ and interior diameter $c_2 h$ with c_1, c_2 independent of the discretization parameter h. This condition means that the sides of the triangles are bounded by ch and the smallest interior angle is bounded below by $\alpha_0 > 0$ independent of h. Furthermore, we require that the intersection of any two triangles is void or consists of a common side or vertex. Now the space of piecewise linear shape functions is defined by

$$S_0^1 = \{v_h \in C^0(\overline{\Omega}) : \text{the restriction } v_h|_{\Lambda_h} \text{ is linear for each triangle } \Lambda_h \in \Pi_h \text{ and } v_h = 0 \text{ on } \partial\Omega\}.$$

Using the definition of weak derivatives we see that each $v_h \in S_0^1$ is differentiable, especially $S_0^1 \subset H_0^{1,2}(\Omega)$.

The finite element method is defined:

$$\text{Find } u_h \in S_0^1 \text{ such that } a(u_h, v_h) = (f, v_h) \quad \forall v_h \in S_0^1. \tag{7}$$

The actual computation of u_h is carried out by the use of the *natural basis* of S_0^1. Let $P_1, \dots, P_N$ be the interior nodal points of the triangulation Π_h. To each $i = 1, \dots, N$ we associate a function $\psi_i \in S_0^1$ by

$$\psi_i(P_j) = \delta_{ij}, \qquad j = 1, \dots, N. \tag{8}$$

Clearly, the set $\{\psi_i\}_{i=1,\dots,N}$ forms a basis of S_0^1 that has the important property that the support of ψ_i consists of the triangles adjacent to P_i. Inserting the expansion $u_h(x) = \sum_{i=1}^N \xi_i \psi_i(x)$, $\xi_i \in \mathbb{R}$, in (7) we obtain that

$$F_j(\xi) = b_j, \qquad j = 1, \dots, N, \tag{9}$$

where

$$F_j(\xi) = a\left(\sum_{i=1}^N \xi_i \psi_i, \psi_j \right), \qquad b_j = (f, \psi_j). \tag{10}$$

It is obvious that the nonlinear system (9) is equivalent to the more abstract definition (7). The system (9) can be solved by Newton's method. We remark that the corresponding linearized system ($A = (a_{ji})$)

$$a_{ji} = (\partial/\partial\xi_i)F_j(\xi)$$

is sparse since

$$a_{ji} \neq 0 \Rightarrow \psi_j, \psi_i \text{ have common support.}$$

For instance, if any node of the triangulation has at most six neighbors then each row and each column of A has not more than seven nonvanishing elements. This explains why piecewise polynomial shape functions are preferred to more classical ansatz functions.

The proof of existence and uniqueness of both (6) and (7) uses the theory of monotone operators. By the ellipticity (2) and the growth condition (4) we obtain that the nonlinear form a is monotone in $H_0^{1,2}(\Omega)$, i.e.,

$$a(u, u-v) - a(v, u-v) \geq \alpha||\nabla(u-v)||^2 \quad \forall\, u, v \in H_0^{1,2}(\Omega). \tag{11}$$

Since (3) guarantees the boundedness of a in $H^{1,2}$, standard arguments show that (6), (7) possess uniquely determined solutions in $H_0^{1,2}(\Omega)$ and S_0^1, respectively.

The monotonicity (11) can also be used for deriving an error estimate. We obtain from (11) and (6), (7) for arbitrary $v_h \in S_0^1$ that

$$\begin{aligned} ||\nabla(u-u_h)||^2 \leq & \frac{1}{\alpha}\{a(u, u-u_h) - a(u_h, u-u_h)\} \\ = & \frac{1}{\alpha}\{a(u, u-v_h) - a(u_h, u-v_h\}. \end{aligned}$$

Now the mean value theorem and the growth condition (4) yield

$$||\nabla(u-u_h)||^2 \leq c||\nabla(u-u_h)||\,||\nabla(u-v_h)||,$$

and hence

$$||\nabla(u-u_h)|| \leq c||\nabla(u-v_h)|| \quad \forall v_h \in S_0^h. \tag{12}$$

Thus, an error estimate is given by inserting a special approximation of u in (12). Using the definition of the natural basis in (8) we define the nodal interpolation $I_h u \in S_0^1$ of u by

$$I_h u(x) = \sum_{i=1}^{N} u(P_i)\psi_i(x).$$

For the interpolation error one can prove that

$$||\nabla(u - I_h u)|| \leq ch||\nabla^2 u||, \tag{13}$$

which implies that the finite element method with piecewise linear shape functions is of first order in the energy norm $||\nabla \cdot ||$ if the solution u is sufficiently smooth.

This approach can be generalized in various respects without essentially changing the arguments demonstrated above. Nevertheless, the book contains a lot of technical materials.

Other finite element spaces. Instead of using piecewise linear shape functions one can improve the accuracy of the scheme by using elements of higher degree. The author presents his own theory on the construction of finite elements lying in the space $C^m(\Omega)$. Furthermore, Zlàmal's ideal elements for nonpolygonal boundaries are described and error estimates similar to (13) are proved. Both topics cannot be found in other books concerning the finite element method.

Numerical cubature. In general it is impossible to compute the integrals in (11) exactly so that numerical cubature has to be used. The author proves conditions on the cubature formula such that the modified discrete solution exists and converges with the same rate as the solution of (7).

Problems with lower regularity. The author also considers piecewise smooth problems, which means that condition (3) only holds in a set of subdomains of Ω. This corresponds to nonsmooth solutions $u \in H^{1+\varepsilon}(\Omega)$ or $u \in H^1(\Omega)$ for which the interpolation is not defined. In this case, one can use smoothing arguments or other approximations for proving convergence.

The last part of the book is devoted to the analysis of the fully discrete finite element method for parabolic and mixed parabolic-elliptic problems in a more abstract setting. The results are similar to those proved in the elliptic case.

Alexander Zenisek's book presents a detailed mathematical and numerical analysis of the conforming finite element method for second order elliptic and parabolic problems. Some programs enable the reader to reproduce the results on his own personal computer. Since the book is well written and contains a chapter with mathematical background material, it can be read with great profit by researchers and engineers as well as beginners. Especially the chapters on the construction of finite element spaces and special approxmations for nonsmooth solutions may be interesting for researchers. This topic is not contained in other books in detail. But we remark that the title of the book is misleading as far as the word *nonlinear* is concerned. The mathematical and numerical treatment of the nonlinear problems introduced above does not essentially differ from that of linear problems such as (5). Due to the restrictive growth and ellipticity condition, typical nonlinear phenomena such as nonuniqueness and bifurcation cannot occur.

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