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BOOK REVIEW

Walsh series and transforms, by Golubov, Efimov, and Skvortsov. Kluwer Academic Publishers, Dordrecht, The Netherlands 1991, 367 pp., \$169.00. ISBN 0-7923-1100-0

An *orthonormal system* on the interval $[0,1)$ is a sequence of functions $\phi_0, \phi_1, \dots$ that satisfies

$$\int_0^1 \phi_k(x)\phi_j(x)\,dx = \begin{cases} 1 & k = j, \\ 0 & k \neq j. \end{cases}$$

The Walsh system occupies a unique position among orthonormal systems on $[0,1)$. It is the completion of the Rademacher system, a prototype for all independent, identically distributed sequences of random variables. It is the simplest complete, orthonormal system: each Walsh function takes only the values ± 1. It can be defined by using products of functions of mean zero (Rademacher functions) and up to measure preserving transformations, it is the only such system on a probability space whose functions have range $\{1, -1\}$ (Waterman [62]). Roughly speaking, the Walsh system is the only real-valued orthonormal system whose functions alternate signs on finer and finer partitions of $[0,1)$ (Levizov [34]) and the only one such that the 2^nth partial sums of its Fourier series form a sequence of positive operators, i.e., if $f \geq 0$ then $S_{2^n} f \geq 0$ for $n = 0, 1, \dots$ (Price [41]).

There are several reasons for studying the Walsh system.

The Walsh system is similar to the trigonometric system but simpler. Garnett and Jones [21] used this to obtain information about classical Hardy space by first proving a result about the Walsh case. Konyagin [32] solved a longstanding problem concerning representation of infinite-valued functions by trigonometric series by proving that a trigonometric series cannot diverge to $+\infty$ on a set of positive measure. (The Walsh analogue of this result had been proved twenty years earlier by Talalyan and Arutunyan [55].) Sjölin [49] obtained one of the best positive results on a.e. convergence of trigonometric series by first dealing with the Walsh case. In particular, some of the more difficult aspects of the trigonometric theory are easier to understand (and might be better taught) in the simpler Walsh case first.

The Walsh system has influenced other areas of mathematics. Enflo [15] used it to construct a separable Banach space that has no basis. This solved a longstanding problem posed by Banach in the twenties. The fundamental theorem of martingale inequalities was first proved for the Walsh system (Paley [40]); and, the Walsh functions were used by Bethke [8] to analyze the behavior of genetic algorithms, a method to optimize nondifferentiable functions which works when the traditional methods either fail to converge or converge badly at suboptimal points.

The Walsh system fills an important niche in the general theory of harmonic analysis on compact abelian groups. Vilenkin [56] showed that given any zero-dimensional, compact abelian metrizable group G there are primes $p_0, p_1, \ldots$ that generate a Walsh-like system that exhausts the characters of G. Thus the Walsh system is a prototype for character systems of zero-dimensional metrizable compact abelian groups.

The Walsh system occupies an even more central position in the theory of harmonic analysis on locally compact fields (see Taibleson [54, p. 5]). Such fields have been mentioned by Volovich [57] as a replacement for string theory as a basis for theoretical physics at the sub-Plank scale.

In the literature, there are two types of systems that are called *multiplicative* and the Walsh system is a prototype for both types. The more general of the two was introduced and developed by the Hungarian school (see Alexits [2] and Schipp [43]) and is defined using product systems. This definition does not require an underlying group structure. The other type comes from a concrete way of constructing certain Vilenkin groups (hence Vilenkin systems) and uses infinite direct products of integer groups of order p_k, $k = 0, 1, \ldots$. Outside the Soviet Union these systems are called *Vilenkin systems* and the corresponding groups are called *Vilenkin groups*. Inside the Soviet Union, these systems are usually referred to as multiplicative systems. To distinguish these from more general multiplicative systems, we prefer to call them *multiplicative Vilenkin systems*. When $p_k = 2$ for $k = 0, 1, \ldots$, the corresponding multiplicative Vilenkin system is precisely the Walsh system and the corresponding group is called the *dyadic group*. A Vilenkin group is said to be of *bounded type* if $\sup p_k < \infty$. Thus the dyadic group is a Vilenkin group of bounded type.

The Walsh system has many applications in the real world. The Fast Walsh Transform is simpler and more efficient than the trigonometric Fast Fourier Transform but can be used in many of the same applications (e.g., filtering, pattern recognition, image enhancement, and multiplication of large numbers). For this reason, in applications where weight must be conserved (e.g., in the Mariner space craft that explored Mars and in some guidance systems for small missiles), the Walsh system is preferred over the trigonometric system. Some sampling techniques from spectroscopy simulate sampling Walsh-Fourier coefficients and in these cases the Walsh system is used to recapture the original signal. There is even some evidence that the two-dimensional Walsh system can be used to decipher the neural code that transfers information from the eye to the brain in higher primates. (A short survey of these topics and pertinent references can be found in Schipp, Wade, and Simon [45].)

J. L. Walsh (1895–1973), an early student of G. D. Birkhoff who had a long and successful career in approximation theory at Harvard where he became a member of the National Academy of Sciences and president of the American Mathematical Society, introduced the Walsh functions in 1923 (see Walsh [61]). Haar [27] had introduced a complete, nonuniformly bounded orthonormal system on the interval $[0, 1)$ with good approximation properties whose functions were piecewise constant. Walsh used linear combinations of Haar functions to produce a complete, uniformly bounded, orthonormal system of piecewise constant functions on $[0, 1)$ whose Fourier series, when attention is restricted to 2^nth partial sums, have the same convergence properties as the Haar system. He proved that the 2^nth partial sums of the Walsh-Fourier series of a function f converges to f uniformly when f is continuous on $[0, 1)$ and a.e. when f is integrable on $[0, 1)$.

Paley [40] gave a new, more tractable definition of the Walsh functions using products of Rademacher functions. His system $w_0, w_1, \dots$, usually called the Walsh-Paley system, coincides with Walsh's system but is a different enumeration. This enumeration, the one used most frequently by mathematicians, is the main topic of the book under review. It retains the advantages of the earlier system (good convergence properties for 2^nth partial sums and close analogy with the trigonometric functions) while adding two new features. By using products of Rademacher functions, Paley provided a vehicle to obtain explicit representations of kernels of certain operators associated with Walsh-Fourier series (for example, the sequence of partial sums) and forged a link between the theory of Walsh series and probability theory.

The *Walsh-Fourier coefficients* of an integrable function f are the numbers

$$\hat{f}(k) := \int_0^1 f(x) w_k(x)\, dx \qquad k = 0, 1, \dots,$$

and the *Walsh-Fourier series* of f is the series

$$Sf := \sum_{k=0}^{\infty} \hat{f}(k) w_k.$$

The central problems in this area are:

The growth problem. Find ways in which properties of f (e.g., smoothness) are reflected and characterized by properties of its Walsh-Fourier coefficients $\{\hat{f}(k)\}$.

The convergence problem. Determine whether a given function f can be approximated (in some sense, e.g., uniformly, in norm, or by some summability technique) by its Walsh-Fourier series Sf.

The uniqueness problem. Identify conditions such that if a Walsh series $S := \sum_{k=0}^{\infty} a_k w_k$ converges to an integrable function f then $S = Sf$, i.e., $a_k = \hat{f}(k)$ for $k = 0, 1, \dots$.

Arguably the most elegant solution to the growth problem was obtained by Bochkarev [9], who characterized the growth of Walsh-Fourier coefficients of nonconstant continuous functions. He proved that if f is continuous and its Walsh-Fourier coefficients satisfy $|\hat{f}(k)| = O(k \log k)^{-\alpha}$ as $k \to \infty$ for some $\alpha > 1$, then f is constant. (This theorem has no trigonometric analogue.) On the other hand, there exist nonconstant continuous functions f that satisfy $|\hat{f}(k)| = O(k \log k)^{-1}$ as $k \to \infty$.

Solutions to the convergence problem include the following. If $f \in L^p[0, 1)$ for some $1 < p < \infty$ then $S_n f \to f$ in L^p norm (Paley [40]). If $f \in L \log^+ L \log \log^+ L$ (in particular, if $f \in L(\log^+ L)^p$ for some $p > 1$) then Sf converges to f a.e. (Sjölin [49]). On the other hand, there exist $f \in L(\log^+ \log^+ L)^p$ for $p < 1$ such that $S_n f$ diverges everywhere (Schipp [44]). It is still not known whether the Walsh-Fourier series of every $f \in L \log^+ L$ converges a.e. or if one can diverge on a set of positive measure.

Convergence properties for continuous functions f are somewhat better: $S_n f \to f$ uniformly if f is of bounded fluctuation (Onneweer and Waterman [39]) or if f belongs to the Lipschitz space of order α for some $\alpha > 0$ (Fine [18]). But, given an $\mathcal{F}_\sigma$ set E of measure zero there is a function f continuous on the dyadic group whose Walsh-Fourier series diverges everywhere on E (Harris [30]). It is an open question whether this result holds for all Borel sets E of measure zero.

A set E is called a *set of uniqueness* if the only Walsh series that converges to 0 off E is the zero series. Solutions to the uniqueness problem include the following. The empty set (Walsh [61]) and every countable set (Fine [18]) is a set of uniqueness, but some sets of measure zero are sets of uniqueness and others are not (Shneĭder [46]). Thus the uniqueness problem cannot be solved for a.e. convergent series unless additional requirements are made. It turns out that, as in the trigonometric case, there is a subtle connection between analytic number theory and the uniqueness problem (see Aubertin [4]). In fact, a set of uniqueness is not determined by its set theoretic properties but by its algebraic and number theoretic properties. A good example of this is that every subgroup of the dyadic group of measure zero is a set of uniqueness (Yoneda [65]). A complete characterization of sets of uniqueness is still a long way off. For example, we do not yet know whether the Cantor (middle thirds) set is a set of uniqueness or not.

More is known about the uniqueness problem when 2^nth partial sums are used and sets of uniqueness for 2^nth partial sums of Walsh series that satisfy certain growth conditions have been characterized. Arutunyan and Talalyan [3] and Crittenden and Shapiro [14] were first to examine this problem. A corollary of their work is that if the 2^nth partial sums of a Walsh series S converge to a finite-valued, integrable function f at all but countably many points on the interval $[0, 1)$, and if S satisfies the $C - S$ growth condition, i.e., if $2^n S_{2^n}(x \pm 0) \to 0$ for every $x \in [0, 1)$, then S is the Walsh-Fourier series of f. Crittenden and Shapiro showed this theorem is best possible in the following sense. A Borel set E is a set of uniqueness for 2^nth partial sums of Walsh series that satisfy the $C - S$ condition if and only if E is countable.

At first glance the theory of Walsh series seems to run along very similar lines as that of trigonometric series. This is true, but most of the deeper results need different proofs and approaches. Moreover, there are many situations where the theory of Walsh series is significantly different from that of trigonometric series. Here are several examples.

1. In general, convergence properties of trigonometric Fourier series are not improved by passing to subsequences of partial sums. For Walsh-Fourier series, convergence properties of 2^nth partial sums are markedly better than those of nth partial sums. For example, if f is continuous then $S_{2^n} f \to f$ uniformly on $[0, 1)$ (Walsh [61]) but $S_n f$ may diverge on a set of measure zero.

2. Trigonometric Fourier series exhibit Gibbs phenomenon, a predictable overshooting of the value of a function near a discontinuity. The Walsh system does not exhibit this phenomenon at dyadic rational points of discontinuity (because the Walsh functions themselves have jump discontinuities there) but does exhibit it at dyadic irrational points of discontinuity and with a larger percentage of overshoot, namely, at lest 1.33 instead of 1.178 (compare Balashov and Skvortsov [6] with Zygmund [68]).

3. For the trigonometric system, the smoother a function the faster its Fourier coefficients converge to zero. In sharp contrast, the Walsh-Fourier coefficients of a smooth nonconstant function cannot decay too rapidly. (For example, see Bochkarev's solution to the growth problem cited above.) This comes about because of a reverse Gibbs phenomenon. Namely, if the Walsh-Fourier coefficients

decay rapidly enough then a discontinuity introduced by Walsh functions of low order cannot be cancelled by the relatively small contribution of higher order terms.

4. The trigonometric system, as the group of characters of the one-dimensional torus, is algebraically ordered. The Walsh system, as the group of characters of the dyadic group, cannot be algebraically ordered. This results in two fundamental differences between the Walsh system and the trigonometric system. First, for the trigonometric system, each function gives rise to a unique conjugate function. In the Walsh case, at least as far as the convergence problem is concerned, a replacement for the conjugate function is provided by martingale transforms, and thus each function gives rise to a class of conjugate functions. (For the connection between conjugate functions and groups whose duals are algebraically ordered, see Rudin [42].) Secondly, the fact that the Walsh character group cannot be algebraically ordered makes a drastic difference in continuity of multipliers in two dimensions. For the trigonometric case, simple domains like the triangle generate multipliers for trigonometric series that are continuous on $L^p[0, 2\pi)$ for all $1 < p < \infty$. For the Walsh case, a triangle generates a continuous multiplier on $L^p[0, 1)$ if and only if $p = 2$ (see [45]). Closely related to this is the fact that the normalized trigonometric system and the Walsh system are equivalent bases in $L^p[0, 1)$ for $1 < p < \infty$ only when $p = 2$ (Wo-Sang Young [66]).

Other differences show up in the general Vilenkin setting. For example, in the trigonometric case, Hardy spaces have an atomic structure (Coifman and Weiss [13]) but for Vilenkin groups G, martingale Hardy spaces and atomic Hardy spaces coincide only when G is of bounded type (Simon [48]). For Vilenkin groups of unbounded type, atomic Hardy spaces are somewhat smaller than martingale Hardy spaces and seem to be the right spaces to use for harmonic analysis (Fridli and Simon [20]).

The modern theory of Walsh functions owes much to the school of Zygmund. In Coifman and Strichartz [12], which lists only first and second generation students of Zygmund, there are no fewer than thirteen mathematicians who took part in the development of this area. The first of these was Fine [18], who showed that the Walsh functions are the characters of a compact abelian group (the dyadic group). This connected Walsh analysis with Fourier analysis on groups, providing algebraic and topological structures for the unit interval in which the Walsh functions are both multiplicative and continuous. As a consequence, analogues of certain trigonometric techniques (e.g., translation, convolution, and the Riesz representation theorem for functionals of "continuous" functions) were available to Walsh analysis for the first time.

The next year Fine [19] extended the domain and index set of Walsh functions to $[0, \infty)$. These generalized Walsh functions play the same role on the interval $[0, \infty)$ as the functions $\{\exp(2\pi ixy) : y \in (-\infty, \infty)\}$ play on $(-\infty, \infty)$. In particular, they can be used to define the Walsh-Fourier transform of a function f integrable on $[0, \infty)$, an analogue of the classical Fourier transform. (There is a similar generalization of Vilenkin systems which generates Vilenkin transforms and locally compact Vilenkin groups.) As Walsh-Fourier series can be used to represent functions integrable on $[0, 1)$, even so, the Walsh transform can be used to represent functions integrable on $[0, \infty)$. In a general sense, the study of Walsh series and transforms is concerned with how faithful these representations are.

During the 1950s others began to show an interest in the Walsh system. Using the foundation laid by Fine and Vilenkin, Yano (e.g., [63], [64]) wrote a series of papers examining the convergence problem from the point of view of summability and Shneĭder (e.g., [46]) wrote a series of papers examining the uniqueness problem. He introduced Riesz products for the Walsh system and used them to obtain an analogue of Rajchman's theorem that gives conditions sufficient for a given set to be a set of uniqueness. (To this day we have no other method for constructing sets of uniqueness.)

The papers of Fine, Vilenkin, Shneĭder, and Yano initiated two decades of intense activity that expanded the scope of the theory in several quite different directions while reinforcing the analogy with trigonometric series. (A survey of this development can be found in [5] and [58].) At the same time, Walsh-Fourier analysis began to take on an identity of its own.

At the heart of this separate identity is a deep connection between Walsh analysis and probability theory. In 1957 Morgenthaler [37] published a central limit theorem for lacunary Walsh series. This shows that a lacunary sequence of Walsh functions behaves much like a sequence of independent, identically distributed random variables. In 1965 Gundy [26] noticed that the 2^nth partial sums of any Walsh series form a martingale. Thus convergence theorems for martingales provide simple explanations for convergence of $S_{2^n}f$ and give necessary and sufficient conditions that the 2^nth partial sums of a nonFourier-Walsh series converge. And in 1966, Burkholder [10] generalized Yano's weak type $(1,1)$ inequality from Walsh series to general martingale transforms.

By the early seventies, Walsh analogues of most of the classical techniques had been developed. However, two central ideas used in the classical theory of trigonometric series still had no Walsh analogues: derivatives and Hardy spaces. The classical derivative is useless for Walsh analysis since each Walsh function is piecewise constant. The classical theory of Hardy spaces relies on interpreting trigonometric series as boundary values of analytic functions. No such interpretation is available for Walsh series.

In 1973 Hardy spaces for Vilenkin groups (hence for Walsh analysis) were provided by probability theory. Fefferman and Stein [17] had shown that real H^p spaces are characterized by integrability conditions on the Littlewood-Paley function. Garsia [22], using the martingale maximal function in place of the Littlewood-Paley function, constructed Hardy spaces on probability spaces. Applying these ideas to the martingale formed by 2^nth partial sums of Walsh-Fourier series, one obtains *dyadic Hardy spaces*, which serve as replacements for the classical Hardy spaces when studying harmonic analysis of the Walsh system. For example, Ladhawala [33] proved a Walsh analogue of Hardy's theorem concerning growth of Fourier coefficients of functions in H^1.

Also in 1973 the *dyadic derivative* was defined by Butzer and Wagner [11], who adapted a discrete version introduced earlier by engineers Gibbs and Millard [23]. This derivative $d^{[1]}$ is a Walsh analogue of the classical derivative in the sense that the Walsh functions are its eigenfunctions with $d^{[1]}w_k = kw_k$ for $k = 0, 1, \ldots$. It satisfies some of the usual properties (e.g., it is a linear operator) but not all. For example, it does not satisfy the chain rule. Moreover, continuity is not weaker than differentiability, even on the dyadic group: there exist strongly dyadically differentiable functions which are not continuous (see Ladhawala [33]). And, the dyadic derivative is not a local operator. Instead, its definition allows values of f at remote points to affect the size of $d^{[1]}f$ at near points. For example, though the functions w_0 and w_1 are identical on the interval $[0, \frac{1}{2})$, their dyadic derivatives satisfy $d^{[1]}w_0(x) = 0 \neq d^{[1]}w_1(x) = 1$ for all $x \in [0, \frac{1}{2})$.

Because of the extreme nonlocal character of the dyadic derivative, classical continuity and dyadic differentiability are incompatible. Indeed, a continuous function cannot be everywhere dyadically differentiable ([52]). Roughly speaking, the only functions which are dyadically differentiable are piecewise constant functions (Engels [16]). However, there is a "dyadic integral" that plays the role of antidifferentiation for $d^{[1]}$. The fundamental theorem of calculus for this integral was obtained by Butzer and Wagner in the strong case and Schipp in the pointwise case. In fact, Schipp proved that if μ is a finite Borel measure then the indefinite dyadic integral of μ is a.e. dyadically differentiable and its derivative is exactly the absolutely continuous part of μ (see [45] for details, a history, and references). Thus Lebesgue's theorem for differentiation has a dyadic analogue.

The dyadic derivative has generated several new problems and has contributed new solutions to growth, convergence, and summability problems. For more information on these developments see the proceedings of a conference held in Yugoslavia, especially the bibliography in Stanković and Gibbs [53].

The 1980s have been a period of consolidation. Results have become increasingly technical and more refined but understanding has increased. Not accidentally, at the same time an assault has begun on new frontiers: general Vilenkin systems and multiple Walsh series and transforms. A recent survey of solutions to the convergence problem can be found in [59]. A more general but less recent survey (up to 1987) appears in [45].

The convergence problem has been recast in terms of martingale convergence and symmetric differentiation. Schipp [43] used nonlinearly ordered martingales to provide a new proof of a.e. and norm convergence of Walsh-Fourier series of functions in $L^p[0,1)$, $p > 1$. The idea is elegant and profound. The maximal function S^* associated with the partial sums of the Walsh-Fourier series of an $f \in L^p[0,1)$ can be estimated using a nonlinearly ordered martingale transform. Burkholder [10] has shown that maximal operators of *linearly ordered* martingale transforms are of type (p,p) for $1 < p < \infty$. By adapting this proof to the nonlinear case Schipp showed directly that S^* is of type (p,p). This proof is simpler and more natural than either Carleson's proof using exceptional sets or Fefferman's proof using trees. Moreover, this method has yielded Sjölin's theorem, the most general result of its kind (see [45] for details), and has been used on a large class of multiplicative systems that includes all Vilenkin systems of bounded type (see Schipp [44]).

In another direction, Skvortsov [50] discovered a connection between convergence of 2^nth partial sums of Walsh series S and symmetric differentiation of the first formal integral of S. Using the classical result that symmetric derivatives either do not exist or are finite a.e. he gave a simple proof of the Talalyan-Arutunyan result that (the 2^nth partial sums of) a Walsh series cannot diverge to $+\infty$ on a set of positive measure.

The uniqueness problem has been recast in terms of measure theory. Riemann had used the second formal integral to study uniqueness of trigonometric series. Fine [18] adapted this technique to Walsh series and found that only the first integral was needed. It was a highly refined tool in Crittenden and Shapiro [14] but involved subtle topological arguments for its application. Yoneda (see [60]) discarded the first integral in favor of a simpler, more Walsh-like technique that has no trigonometric analogue: quasi-measures. (A *quasi-measure* is a set function on the collection of dyadic intervals which is finitely additive.) It turns out that *every* Walsh series is the Walsh-Fourier-Stieltjes series of some quasi-measure and limits of 2^nth partial sums of Walsh series are essentially symmetric derivatives of quasi-measures. Hence the uniqueness problem reduces to determining when a quasi-measure is a Borel measure that has no singular part. This offers a simple explanation of a corollary to the Arutunyan-Talalyan and Crittenden-Shapiro uniqueness result. Convergence of S_{2^n} to zero at all but countably many points can be interpreted as a differentiability condition on quasi-measures and the $C-S$ condition can be interpreted as a continuity condition. Since the only quasi-measure that is differentiable (with derivative zero) and continuous (i.e., with no point masses) is the zero measure, it is not difficult to see that S must be the zero series. This striking point of view is due to Grubb [25] who greatly simplified the study of uniqueness of 2^nth partial sums and extended these results to the general Vilenkin case. In fact, this approach can be used in situations where no underlying group structure exists (Grubb [24]).

Walsh analysis has reached sufficient maturity that several books have been written. Harmuth [28], [29], Beauchamp [7], Maqusi [35], and Zalmanzon [67] are engineering books that describe applications but include little or no mathematical theory.

Taibleson [54], Agaev, Vilenkin, Dzafarli, and Rubinshteĭn [1], Siddiqi [47], Schipp, Wade, and Simon [45], and the book under review are theoretical mathematical texts, designed to introduce a beginning graduate student to the basic problems in the area. [54] and [1] deal with Vilenkin systems; the former emphasizes the role the field structure plays, thus concentrating on transforms, while the later emphasizes the compact case, thus concentrating on series. [47] and [45] deal almost exclusively with the Walsh case, the former being a short tract on Walsh-Fourier series that examines only elementary solutions to the convergence problem, the latter being a long, comprehensive monograph on Walsh series and Walsh transforms designed for specialists with some background in functional analysis and probability theory.

The book under review falls somewhere in between and appeals both to engineers and mathematicians. It is a readable text of value as an introduction to the main themes of Walsh-Fourier analysis. It gives a thorough treatment of the classical theory and of applications from the authors' points of view. It contains a short section of historical notes and a six page bibliography. It also contains appendices dealing with elementary material (abelian groups, Lebesgue measure on the real line, the abstract Lebesgue integral, and Hilbert spaces). Thus it strives to be readable to anyone with an undergraduate background in mathematics. The specialist may grow impatient, but engineers and physicists will appreciate the authors' care in gently introducing concepts like weak type (p, p) operators.

This book is translated from the russian by a mathematician (the reviewer). Thus, unlike some translations of mathematics books at this level, it contains mathematically correct statements of theorems and definitions and uses standard, current mathematical jargon. However, there are some places where the translation is too literal (resulting in awkward English) and a couple of places where grammatical errors still survive (a singular verb with a plural subject and a violation of the sequence of tenses rule).

This book has three main parts.

The first part (Chapters 1 through 10, excluding Chapter 6) is devoted to the classical theory of Walsh series and Walsh-Fourier series. The results here are nicely balanced, including several solutions to the convergence, uniqueness, and growth problems. These chapters contain occasional short sections on multiplicative Vilenkin systems.

The second part (Chapter 6) gives a brief introduction to Fourier transforms generated by multiplicative Vilenkin systems. Included here is the Hausdorff-Young inequality, the Plancherel transform, its extension to $L^p[0,\infty)$ for $1 < p < 2$, and several variants of the inversion theorem .

The third part (Chapters 11 and 12) gives a thorough introduction to the use of multiplicative Vilenkin systems and transforms for digital information processing, including information compression, fast algorithms, digital filtering, holographic transforms, and optimization. This part contains some new material.

Some topics have been omitted entirely. The chapter on transforms does not mention the Mellin transform nor does it contain any summability results for inversion of the Walsh-Fourier transform. There is no mention of the dyadic derivative. Although Morgenthaler's theorem is included, there is no mention of the deeper connections between the Walsh system and probability theory. Quasi-measures are not mentioned, but there are several sections on the symmetric derivative of functions that provide an introduction to this relatively late development and make the connection between differentiation and the convergence problem crystal clear.

In spite of these omissions, this text includes many classical solutions to the fundamental problems. It contains Moon's proof [36] that everywhere divergent Walsh-Fourier series exist, Hunt's proof [31] that Walsh-Fourier series of an $f \in L^2[0,1)$ converges a.e., a uniqueness result of Skvortsov [51] that uses a growth condition more general than the $C - S$ condition, proofs that Walsh-Fourier series are summable and converge in norm, special results about Walsh series with monotone coefficients, about the rate of approximation by Walsh series and by multiplicative Vilenkin systems, and many other results. Moreover, the more technical results from the general theory (e.g., Bochkarev's characterization of the growth of Walsh-Fourier coefficients of continuous functions, Onneweer and Waterman's convergence theorem for functions of bounded fluctuation, and Sjölin's a.e. convergence theorem) are mentioned in the historical notes or in the text itself. Therefore, the reader gets a broad introduction to the study of Walsh functions. In fact, the book accomplishes its goal as stated in its preface: to give technical specialists and mathematicians "an account of the theoretical foundations of Walsh series and Walsh transforms" and "access to the literature on Walsh series."

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