

**A SHARP POINTWISE BOUND FOR FUNCTIONS  
WITH  $L^2$ -LAPLACIANS ON ARBITRARY DOMAINS  
AND ITS APPLICATIONS**

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ABSTRACT. For all functions on an arbitrary open set  $\Omega \subset R^3$  with zero boundary values, we prove the optimal bound

$$\sup_{\Omega} |u| \leq (2\pi)^{-1/2} \left( \int_{\Omega} |\nabla u|^2 dx \int_{\Omega} |\Delta u|^2 dx \right)^{1/4}.$$

The method of proof is elementary and admits generalizations. The inequality is applied to establish an existence theorem for the Burgers equation.

1. INTRODUCTION

In this note we announce the proof of the inequality

$$(1) \quad \sup_{\Omega} |u| \leq \frac{1}{\sqrt{2\pi}} \left( \int_{\Omega} |\nabla u|^2 dx \int_{\Omega} |\Delta u|^2 dx \right)^{1/4}$$

for functions with zero boundary values on three-dimensional domains. The domain  $\Omega$  can be any open set and the constant  $1/\sqrt{2\pi}$  is optimal. This best possible result is obtained by a new and elementary method, which is apparently also applicable to other elliptic operators. Thus, many known inequalities can be improved and new ones derived. Some of these will be given by the author in separate papers. Such inequalities are used in the study of nonlinear differential equations, see [1] and [2].

For smoothly bounded domains, one can combine the Sobolev inequality (see [3])

$$\sup_{\Omega} |u| \leq C_1(\Omega) \|\nabla u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2}$$

with the a priori estimate (see [4])

$$(2) \quad \|u\|_{H^2(\Omega)} \leq C_2(\Omega) \|\Delta u\|_{L^2(\Omega)}$$

to obtain

$$(3) \quad \sup_{\Omega} |u| \leq C_3(\Omega) \|\nabla u\|_{L^2(\Omega)}^{1/2} \|\Delta u\|_{L^2(\Omega)}^{1/2},$$

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where  $C_i(\Omega)$  are constants depending on the domain  $\Omega$ . However, the elliptic estimate (2) fails to hold for domains with reentrant corners [5].

It was suggested to the author by Professor J. G. Heywood that (3) should be valid for nonsmooth domains as well, and its generalization to the Stokes operator would yield results for the Navier-Stokes equations in nonsmooth domains. Here, in §4, we use (1) to derive a priori estimates and prove an existence theorem for the initial-boundary value problem of the Burgers equation with  $H^1$  initial data, in an arbitrary open set, for the first time.

## 2. THE MAIN RESULTS

Let  $\Omega$  be an arbitrary open set in  $R^3$ . Let  $\|\cdot\|$  denote the  $L^2(\Omega)$  norm. The homogeneous Sobolev space  $\widehat{H}_0^1(\Omega)$  is defined to be the completion of  $C_0^\infty(\Omega)$  in the Dirichlet norm  $\|\nabla \cdot\|$ , where  $\nabla$  is the gradient. Let  $\Delta$  denote the Laplacian in the sense of distributions. Our main result is

**Theorem 1.** *For all  $u \in \widehat{H}_0^1(\Omega)$  with  $\Delta u \in L^2(\Omega)$ , there holds*

$$\sup_{\Omega} |u| \leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}.$$

*The constant  $1/\sqrt{2\pi}$  is optimal for each  $\Omega$ .*

The space  $\widehat{H}_0^1(\Omega)$  contains the standard Sobolev space  $H_0^1(\Omega)$ . It contains functions that are not square integrable for some unbounded domains. If  $u \in H_0^1(\Omega)$ , then using  $\|\nabla u\|^2 = -\int_{\Omega} u \Delta u \, dx \leq \|u\| \|\Delta u\|$ , we obtain

**Corollary 1.** *If  $u \in H_0^1(\Omega)$  and  $\Delta u \in L^2(\Omega)$ , then  $u$  also satisfies*

$$\sup_{\Omega} |u| \leq \frac{1}{\sqrt{2\pi}} \|u\|^{1/4} \|\Delta u\|^{3/4}.$$

In particular, we obtain a pointwise bound for any normalized eigenfunction of the Laplacian, in terms of its corresponding eigenvalue.

**Corollary 2.** *If  $u$  satisfies*

$$-\Delta u = \lambda u, \quad u \in H_0^1(\Omega), \quad \|u\| = 1,$$

*then*

$$\sup_{\Omega} |u| \leq \frac{\lambda^{3/4}}{\sqrt{2\pi}}.$$

All of the above results are also valid for vector-valued or complex-valued functions. The constants in the corollaries, however, are not optimal.

## 3. OUTLINE OF PROOF

The proof of (1) has four steps.

*Step 1.* First we assume that  $\Omega$  is bounded, with a  $C^\infty$  boundary  $\partial\Omega$ . It is well known that there exist eigenfunctions  $\{\phi_n\}$  of the Laplacian that form a complete orthonormal basis of  $L^2(\Omega)$ , satisfying

$$-\Delta \phi_n = \lambda_n \phi_n, \quad \phi_n|_{\partial\Omega} = 0,$$

where  $\lambda_n > 0$  are the eigenvalues,  $n = 1, 2, \dots$

Let  $x_0 \in \Omega$  and  $m \geq 1$  be fixed. For functions of the form  $u(x) = \sum_{n=1}^m c_n \phi_n(x)$ , we have

$$\frac{u^2(x_0)}{\|\nabla u\| \|\Delta u\|} = \frac{\left(\sum_{n=1}^m c_n \phi_n(x_0)\right)^2}{\left(\sum_{n=1}^m \lambda_n c_n^2\right)^{1/2} \left(\sum_{n=1}^m \lambda_n^2 c_n^2\right)^{1/2}}.$$

This quotient is a smooth and homogeneous function of  $(c_1, \dots, c_m)$  in  $\mathbf{R}^m \setminus \{0\}$ . Hence, at some point  $(\tilde{c}_1, \dots, \tilde{c}_m)$ , it attains its maximum value. The maximum value can be written as

$$4\sqrt{\mu} \sum_{n=1}^m \left(\frac{\phi_n(x_0)}{\mu + \lambda_n}\right)^2,$$

where  $\mu = \sum_{n=1}^m \lambda_n^2 \tilde{c}_n^2 / \sum_{n=1}^m \lambda_n \tilde{c}_n^2$ .

*Step 2.* We introduce the Green function  $G(x; x_0, \mu)$  for the Helmholtz equation

$$\Delta G = \mu G - \delta(x - x_0), \quad G|_{\partial\Omega} = 0.$$

By the maximum principle, we have

$$0 \leq G(x; x_0, \mu) \leq \frac{e^{-\sqrt{\mu}|x-x_0|}}{4\pi|x-x_0|}, \quad \forall x \in \Omega \setminus \{x_0\},$$

the upper bound being the fundamental solution. Hence

$$\int_{\Omega} G^2 dx \leq \int_0^{\infty} \left(\frac{e^{-\sqrt{\mu}r}}{4\pi r}\right)^2 4\pi r^2 dr = \frac{1}{8\pi\sqrt{\mu}}.$$

By Parseval's equality and Green's formula, we have

$$\int_{\Omega} G^2 dx = \sum_{n=1}^{\infty} \left(\int_{\Omega} G \phi_n dx\right)^2 = \sum_{n=1}^{\infty} \left(\frac{\phi_n(x_0)}{\mu + \lambda_n}\right)^2.$$

Therefore

$$\frac{u^2(x_0)}{\|\nabla u\| \|\Delta u\|} \leq 4\sqrt{\mu} \sum_{n=1}^m \left(\frac{\phi_n(x_0)}{\mu + \lambda_n}\right)^2 \leq 4\sqrt{\mu} \int_{\Omega} G^2 dx \leq \frac{1}{2\pi}.$$

Thus, (1) is true for any function of the form  $u(x) = \sum_{n=1}^m c_n \phi_n(x)$ .

*Step 3.* Now, let  $u$  be any function in  $\widehat{H}_0^1(\Omega)$  such that  $\Delta u \in L^2(\Omega)$ . Let  $u_n$  be the projection of  $u$  in  $\text{span}\{\phi_1, \dots, \phi_n\}$ . We have  $\|\nabla u_n\| \leq \|\nabla u\|$ ,  $\|\Delta u_n\| \leq \|\Delta u\|$ , and  $\lim_{n \rightarrow \infty} u_n = u$  in  $L^2(\Omega)$ . It follows that (1) remains valid.

*Step 4.* Now we proceed to prove Theorem 1. We can choose a sequence of bounded domains  $\Omega_n$  with smooth boundaries such that  $\Omega_1 \subset \Omega_2 \subset \dots$  and  $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ . For each  $n \geq 1$ , there exists a unique  $u_n \in \widehat{H}_0^1(\Omega_n)$  such that

$$\int_{\Omega_n} \nabla u_n \cdot \nabla v dx = \int_{\Omega_n} \nabla u \cdot \nabla v dx, \quad \forall v \in \widehat{H}_0^1(\Omega_n),$$

by the Riesz representation theorem. From this we obtain  $\|\nabla u_n\|_{L_2(\Omega_n)} \leq \|\nabla u\|$ ,  $\Delta u_n = \Delta u|_{\Omega_n}$ , and  $\lim_{n \rightarrow \infty} u_n = u$  in  $\widehat{H}_0^1(\Omega)$ , hence in  $L^6(\Omega)$ . Therefore the proof of (1) is completed.

Let  $u(x) = (1 - e^{-|x|})/|x|$ ; then we have

$$\sup_{x \in \mathbf{R}^3} |u(x)| = 1, \quad \int_{\mathbf{R}^3} |\nabla u|^2 dx = 2\pi, \quad \int_{\mathbf{R}^3} |\Delta u|^2 dx = 2\pi.$$

Hence the equality in (1) holds for  $u$ . By cutting-off  $u$ , we explicitly construct a sequence of functions  $u_n$  with compact support such that

$$u_n(0) \rightarrow 1, \quad \int_{\mathbf{R}^3} |\nabla u_n|^2 dx \rightarrow 2\pi, \quad \int_{\mathbf{R}^3} |\Delta u_n|^2 dx \rightarrow 2\pi,$$

as  $n \rightarrow \infty$ . Given any open set  $\Omega$ , by scaling, we obtain a new sequence of functions with compact support in  $\Omega$ . Since the product

$$\|\nabla u_n\|_{L^2(\mathbf{R}^3)}^{1/2} \|\Delta u_n\|_{L^2(\mathbf{R}^3)}^{1/2}$$

is scale invariant, it is seen that the constant  $1/\sqrt{2\pi}$  in (1) is the best possible.

#### 4. APPLICATION TO THE BURGERS EQUATION

The time-dependent Burgers equation

$$(4) \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u}$$

is sometimes studied for its analogy with the Navier-Stokes equations, with the three-dimensional vector-valued function  $\mathbf{u}(x, t)$  representing the velocity field and the positive constant  $\nu$  the viscosity coefficient. We consider as spatial domain an arbitrary open set in  $\mathbf{R}^3$  and seek  $\mathbf{u}$  that vanishes on the boundary and takes an initial value  $\mathbf{u}_0 \in \widehat{H}_0^1(\Omega)^3$ . From (4) and the vector version of (1), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|^2 + \nu \|\Delta \mathbf{u}\|^2 &= \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} dx \\ &\leq \sup |\mathbf{u}| \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| \\ &\leq \frac{1}{\sqrt{2\pi}} \|\nabla \mathbf{u}\|^{3/2} \|\Delta \mathbf{u}\|^{3/2}. \end{aligned}$$

By using Young's inequality and a comparison theorem, we obtain

$$\|\nabla \mathbf{u}(t)\|^2 \leq \frac{\|\nabla \mathbf{u}_0\|^2}{\sqrt{1-t/T}},$$

and

$$\int_0^t \|\Delta \mathbf{u}(s)\|^2 ds \leq \frac{\|\nabla \mathbf{u}_0\|^2}{2\nu \left(1 - \sqrt[6]{t/T}\right) \sqrt{1 - \sqrt{t/T}}},$$

for  $0 \leq t < T$ , where

$$T = \frac{256\pi^2\nu^3}{27\|\nabla \mathbf{u}_0\|^4}.$$

Beginning with these a priori estimates, and using the methods of [1] and [6], the following theorem is established [7].

**Theorem 2.** For any open  $\Omega \subset \mathbf{R}^3$  and any  $\mathbf{u}_0 \in \widehat{H}_0^1(\Omega)^3$ , there exists a unique function

$$\mathbf{u} \in C([0, T], \widehat{H}_0^1(\Omega))^3 \cap C^\infty(\Omega \times (0, T))^3 \cap C^\infty((0, T), L_\infty(\Omega))^3,$$

satisfying the Burgers equation (4) and taking the initial value  $\mathbf{u}_0$ .

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