# A QUASI-POLYNOMIAL BOUND FOR THE DIAMETER
# OF GRAPHS OF POLYHEDRA

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ABSTRACT. The diameter of the graph of a $d$-dimensional polyhedron with $n$ facets is at most $n^{\log d+2}$

Let $P$ be a convex polyhedron. The *graph* of $P$ denoted by $G(P)$ is an abstract graph whose vertices are the extreme points of $P$ and two vertices $u$ and $v$ are adjacent if the interval $[v, u]$ is an extreme edge (= 1-dimensional face) of $P$. The diameter of the graph of $P$ is denoted by $\delta(P)$.

Let $\Delta(d, n)$ be the maximal diameter of the graphs of $d$-dimensional polyhedra $P$ with $n$ facets. (A facet is a $(d-1)$-dimensional face.) Thus, $P$ is the set of solutions of $n$ linear inequalities in $d$ variables. It is an old standing problem to determine the behavior of the function $\Delta(d, n)$. The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for Dantzig's simplex algorithm for linear programming with any pivot rule.

In 1957 Hirsch conjectured [2] that $\Delta(d, n) \leq n - d$. Klee and Walkup [6] showed that the Hirsch conjecture is false for unbounded polyhedra. They proved that for $n \geq 2d$, $\Delta(d, n) \geq n - d + [d/5]$. This is the best known lower bound for $\Delta(d, n)$. The statement of the Hirsch conjecture for bounded polyhedra is still open. For a recent survey on the Hirsch conjecture and its relatives, see [5].

In 1967 Barnette proved [1, 3] that $\Delta(d, n) \leq n3^{d-3}$. An improved upper bound, $\Delta(d, n) \leq n2^{d-3}$, was proved in 1970 by Larman [7]. Barnette's and Larman's bounds are linear in $n$ but exponential in the dimension $d$. In 1990 the first author [4] proved a subexponential bound $\Delta(d, n) \leq 2^{\sqrt{(n-d)\log(n-d)}}$.

The purpose of this paper is to announce and to give a complete proof of a quasi-polynomial upper bound for $\Delta(d, n)$. Such a bound was proved by the first author in March 1991. The proof presented here is a substantial simplification that was subsequently found by the second author. See [4] for the original proof and related results. The existence of a polynomial (or even linear) upper bound for $\Delta(d, n)$ is still open. Recently, the first author found a randomized pivot rule for linear programming which requires an expected $n^{4\sqrt{d}}$ (or less) arithmetic operations for every linear programming problem with $d$ variables and $n$ constraints.

**Theorem 1.**

$$(1) \qquad\qquad \Delta(d, n) \leq n^{\log d+2}.$$

*Proof.* Let $P$ be a $d$-dimensional polyhedron with $n$ facets, and let $v$ and $u$ be two vertices of $P$. Let $k_v$ $[k_u]$ be the maximal positive number such that the union of all vertices in all paths in $G(P)$ starting from $v$ $[u]$ of length at most $k_v$ $[k_u]$ are incident to at most $n/2$ facets. Clearly, there is a facet $F$ of $P$ so that we can reach $F$ by a path of length $k_v + 1$ from $v$ and a path of length $k_u + 1$ from $u$. We claim now that $k_v \leq \Delta(d, [n/2])$. Indeed, let $Q$ be the polyhedron obtained from $P$ by ignoring all the inequalities that correspond to facets that cannot be reached from $v$ by a path of length at most $k_v$. Let $\omega$ be a vertex in $G(P)$ whose distance from $v$ is $k_v$. We claim that the distance of $\omega$ from $v$ in $G(Q)$ is also $k_v$. To see this consider the shortest path between $v$ and $\omega$ in $G(Q)$. If the length of this path is smaller than $k_v$ there must be an edge in the path that is not an edge of $P$. Consider the first such edge $E$. Since $E$ is not an edge of $P$, it intersects a hyperplane $H$ that corresponds to one of the inequalities that was ignored. This gives a path in $P$ of length smaller than $k_v$ from $v$ to the facet of $P$ determined by $H$, which is a contradiction.

We obtained that the distance from $v$ to $u$ is at most $\Delta(d - 1, n - 1)$ $+ 2\Delta(d, [n/2]) + 2$. This gives the inequality $\Delta(d, n) \leq \Delta(d - 1, n - 1) + 2\Delta(d, [n/2]) + 2$, which implies the statement of the theorem.

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