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ONE CANNOT HEAR THE SHAPE OF A DRUM

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ABSTRACT. We use an extension of Sunada's theorem to construct a nonisometric pair of isospectral simply connected domains in the Euclidean plane, thus answering negatively Kac's question, "can one hear the shape of a drum?" In order to construct simply connected examples, we exploit the observation that an orbifold whose underlying space is a simply connected manifold with boundary need not be simply connected as an orbifold.

1. KAC'S QUESTION

Let (M, g) be a compact Riemannian manifold with boundary. Then M has a Laplace operator Δ , defined by $\Delta(f) = -\operatorname{div}(\operatorname{grad} f)$, that acts on smooth functions on M . The *spectrum* of M is the sequence of eigenvalues of Δ . Two Riemannian manifolds are *isospectral* if their spectra coincide (counting multiplicities). A natural question concerning the interplay of analysis and geometry is: must two isospectral Riemannian manifolds actually be isometric? (When M has nonempty boundary, one can consider the *Dirichlet spectrum*, i.e., the spectrum of Δ acting on smooth functions that vanish on the boundary, or the *Neumann spectrum*, that of Δ acting on functions with vanishing normal derivative at the boundary.) If M is a domain in the Euclidean plane then the Dirichlet eigenvalues of Δ are essentially the frequencies produced by a drumhead shaped like M , so the question has been phrased by Bers and Kac [16] (the latter attributes the problem to Bochner) as "can one hear the shape of a drum?" We answer this question negatively by constructing a pair of nonisometric simply connected plane domains that have both the same Dirichlet spectra and the same Neumann spectra. The domains are depicted in Figure 1.

The simple idea exploited here also permits us to construct the following: (1) a pair of isospectral flat surfaces (with boundary) one of which has a unit-length closed geodesic while the other has only a unit-length closed billiard trajectory; (2) a pair of isospectral potentials for the Schrödinger operator on

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FIGURE 1

a plane domain (using a technique of Brooks [5]); (3) a pair of isospectral, nonisometric domains in the hyperbolic plane; and (4) a pair of isospectral, nonisometric domains in the 2-sphere.

Weyl's proof [23] that the area of a plane domain is determined by the spectrum led to speculation that perhaps the shape of a plane domain (or more generally, of a Riemannian manifold) is audible. The latter was refuted by Milnor [17], who exhibited a pair of isospectral, nonisometric 16-dimensional tori. Other examples followed, including (among others) isospectral pairs of Riemann surfaces constructed by Vignéras [22], Buser [7, 8], Brooks [4], and Brooks-Tse [6]; pairs of lens spaces produced by Ikeda [15]; pairs of domains in \mathbf{R}^4 , due to Urakawa [21]; and continuous families of isospectral metrics on solvmanifolds constructed by Gordon-Wilson [14] and DeTurck-Gordon [11, 12]. However, Kac's question concerning plane domains has remained open.

As will be clear from the discussion of Sunada's theorem below, most known pairs of isospectral manifolds have a common Riemannian cover. Thus it is also of interest to exhibit simply connected isospectral manifolds.

2. SUNADA'S THEOREM

Although the early examples of isospectral manifolds seemed rather *ad hoc*, a coherent explanation for most of them has since been provided. Sunada [19] introduced a general method for constructing pairs of isospectral manifolds with a common finite covering:

Theorem (Sunada). *Let M be a Riemannian manifold upon which a finite group G acts by isometries; let H and K be subgroups of G that act freely. Suppose that H and K are almost conjugate, i.e., there is a bijection $f: H \rightarrow K$ carrying every element h of H to an element $f(h)$ of K that is conjugate in G to h . Then the quotient manifolds $M_1 = H \backslash M$ and $M_2 = K \backslash M$ are isospectral.*

Choosing conjugate subgroups in the above theorem yields isometric manifolds, so one seeks a finite group with a pair of almost conjugate but nonconjugate subgroups. The algebraic condition can be restated as: the representations $L^2(H \backslash G)$ and $L^2(K \backslash G)$ are unitarily equivalent, although $H \backslash G$ and $K \backslash G$ are inequivalent as G -sets.

Bérard [1] gave a new proof of Sunada's theorem, by noting the following:

Proposition. *Let G be a group with subgroups H and K , and suppose that $T: L^2(H \backslash G) \rightarrow L^2(K \backslash G)$ is a unitary intertwining operator. Let V be a Hilbert space on which*

G acts unitarily. Then T induces an isometry $V^H \rightarrow V^K$. (Here V^H denotes the subspace of V consisting of the H -fixed points.)

Note that if G acts by isometries on M and hence on $V = L^2(M)$, then $V^H = L^2(H \backslash M)$ since the H -invariant functions on M are precisely those functions that descend to the quotient manifold, and likewise $V^K = L^2(K \backslash M)$. Bérard then used the proposition along with a variational characterization of eigenvalues of the Laplacian to recover Sunada's theorem. He also pointed out that the assumption that H and K act freely is not necessary. In this case the quotients $H \backslash M$ and $K \backslash M$ are orbifolds, as discussed below, although their underlying spaces may be manifolds with boundary.

3. ORBIFOLDS

An *orbifold* is a space locally modelled on the orbit space of a finite group acting on \mathbf{R}^n ; for a precise definition, see [20] or [18]. In particular, the quotient space $O = G \backslash M$ of a manifold M by a group G acting properly discontinuously is an orbifold. If G acts freely, then O is a manifold; otherwise O may have a singular set arising from fixed points of the action of G .

There are modified definitions of the fundamental group of an orbifold and of orbifold covering maps. The important feature for our purposes is that the underlying space $|O|$ of an orbifold O may have no ordinary proper covering spaces, although O has proper coverings in the orbifold sense; thus an orbifold with a simply connected underlying space need not be simply connected as an orbifold.

Viewing G -invariant functions on M as functions on $G \backslash M$, one defines the spectrum of a quotient orbifold $G \backslash M$ as the eigenvalue spectrum of the Laplace operator acting on the space $L^2(M)^G$ of G -invariant functions on M . In particular, if G is a group that acts by isometries on a Riemannian manifold M and if H, K are almost conjugate subgroups of G , perhaps acting with fixed points, then from the above discussion one obtains isospectral orbifolds $O_1 = H \backslash M$ and $O_2 = K \backslash M$.

4. CONSTRUCTION OF ISOSPECTRAL SIMPLY CONNECTED MANIFOLDS

We now utilize the above observations to construct isospectral simply connected plane domains. We use the discussion of §2 to produce an isospectral pair O_1, O_2 of 2-orbifolds with boundary by modifying a construction due to Buser [8] of an isospectral pair M_1, M_2 of flat surfaces with boundary. Buser's surfaces are constructed as covers of a bordered surface M_0 using a pair of almost conjugate subgroups of $G = \mathrm{SL}_3(\mathbf{F}_2)$ and a representation of $\pi_1(M_0)$ in G , and our orbifolds are similarly constructed as covers of an orbifold O_0 , using the orbifold notion of fundamental group, the corresponding theory of orbifold coverings, and a representation of $\pi_1(O_0)$ in a split extension of G by $\mathbf{Z}/2\mathbf{Z}$; indeed, the orbifold O_i is the quotient by an involutive isometry of Buser's manifold M_i , $i = 0, 1, 2$. We observe that the Neumann orbifold spectrum of O_i is precisely the Neumann spectrum of the underlying manifold $|O_i|$; thus the underlying spaces $|O_1|$ and $|O_2|$ are Neumann-isospectral manifolds with boundary. O_1 and O_2 have a common cover in the orbifold sense, but not in the usual sense; this common cover O is the quotient by an involutive isometry of a common cover M of Buser's surfaces M_1 and M_2 . The underlying spaces $|O_1|$ and $|O_2|$ are simply connected plane domains. We deduce the Dirichlet isospectrality of $|O_1|$ and $|O_2|$ by exploiting the Dirichlet

isospectrality of the double covers M_1 and M_2 , which corresponds to isospectrality of the plane domains with mixed boundary conditions: Dirichlet conditions on the orbifold boundary and Neumann conditions on the singular set; this observation, together with the obvious decomposition of any eigenfunction into an involution-invariant eigenfunction and an involution-anti-invariant eigenfunction establishes the Dirichlet isospectrality. Details will appear elsewhere.

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