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BOOK REVIEW

Degeneration of Abelian Varieties, by Gerd Faltings and Ching-Li Chai. Ergeb. Math. Grenzgeb. (3), vol. 22, Springer-Verlag, New York, 1990, 316 pp., \$39.80. ISBN 3-540-52015-5

The publication of this book was an event long awaited by specialists. The algebraic compactification of the moduli of principally polarized abelian varieties is the culmination of a long and fruitful line of research in algebraic geometry. Before turning to the contribution of the book proper, we will summarize some of the principal events in the history of the subject.

Abelian varieties begin with elliptic curves. An elliptic curve over an algebraically closed field k is determined, up to nonunique isomorphism, by its j-invariant, which can be an arbitrary element of k. One says that the affine line over $\mathbf{Z}$, with coordinate j, is a *moduli variety* for elliptic curves: the word "moduli" here means that j is a parameter. This moduli variety has a natural compactification: the projective line over $\mathbf{Z}$. As j tends to infinity, the corresponding elliptic curve degenerates to a singular cubic. Among singular cubics, the least degenerate are those with an ordinary double point. It is therefore natural to complete the modular picture by making the singular cubics correspond to the exceptional value $j = \infty$. We will come back to this problem further on. For the moment, let us say that the contribution of Chai and Faltings is to extend this type of construction to higher dimensional abelian varieties.

Abelian varieties of arbitrary dimension g were initially studied over the complex field. They may be realized as quotients V/Λ, where $\Lambda \approx \mathbf{Z}^{2g}$ is a lattice in the complex vector space V. The quotient V/Λ is a priori a compact complex torus; it is algebraic (i.e., an abelian variety) if and only if there exists a positive hermitian nondegenerate form on V whose imaginary part takes integral values on Λ. (Such a form gives a polarization of V/Λ.) This description of abelian varieties leads to a transcendental construction of the moduli variety for principal polarized abelian varieties of dimension g over $\mathbf{C}$ as the quotient of the Siegel space S_g by the action of the symplectic group $\mathrm{Sp}(2g, \mathbf{Z})$. This quotient has a natural algebraic compactification: the projective variety associated with the graded ring of Siegel modular forms [3]. The compactification obtained in this manner is *minimal* in the language of Chai and Faltings. The set of points at infinity can be stratified, with each stratum representing the space of principally polarized abelian varieties of a given dimension $< g$. One of the goals of Chai-Faltings is to extend this compactification, which is essentially transcendental, into an algebraic construction over $\mathbf{Z}$.

To construct algebraically a modular variety for principally polarized abelian varieties, one can no longer rely on complex uniformization and periods. On the other hand, we can use the polarization to embed the abelian varieties into a projective space and use a suitable Hilbert scheme H to parametrize the resulting subvarieties.

Before saying more about this construction, we should explain why one works with *polarized* abelian varieties, rather than abelian varieties with no additional structure. The necessity of including a polarization is hidden in the case of elliptic curves (which are canonically polarized), but appears clearly for abelian varieties of dimension two or higher. Indeed, to ensure that the Hilbert scheme H is of finite type, we must bound the degree of the polarization; further, the group of automorphisms of a polarized abelian variety is finite, whereas the group of automorphisms of an abelian variety can be infinite. As we will see below, the presence of automorphisms is a great bother when we treat moduli questions.

Returning to the algebraic construction, we note that the desired modular variety appears as a quotient of H by the group of automorphisms of the ambient projective space. The construction of quotients in algebraic geometry had long been an obstacle. However, Mumford's study of Geometric Invariant Theory [7] provides useful sufficient conditions (involving the crucial notion of "stability") for the construction of the quotient of a variety by an action of a reductive algebraic group. Mumford's theory was initially valid only in characteristic zero, since it relied on the semisimplicity of representations of reductive groups, but the theory was extended 10 years later through the work of Haboush. In any case, via a detailed direct study, Mumford was able to construct quasi-projective modular varieties over $\mathbf{Z}$ corresponding to the following two related moduli problems: the classification of abelian varieties of dimension g, furnished with a polarization of degree d, and the classification of smooth projective connected curves of genus g.

An alternative construction of the modular variety was proposed by Mumford [8] soon after his work on Geometric Invariant Theory. This second construction is based on theta functions, which appear classically (over $\mathbf{C}$) when one writes equations for ample divisors on an abelian variety, working in the universal covering space of the abelian variety. The theory of "theta-null Werte" permits one to find modular parameters for the abelian variety. Mumford showed that the essential information carried by the theta functions may be read from the restrictions of these functions to 2-power division points of the abelian variety. This observation leads to an algebraic approach to theta functions, valid in characteristic $\neq 2$. One emerges with a second construction of modular varieties and their minimal compactifications. Thus, already in 1967 one had a satisfactory theory of moduli, at least in characteristic different from 2.

These modular varieties are "coarse moduli schemes." In other words, they are schemes whose points with values in every algebraically closed field k classify, up to isomorphism, the objects being studied over k. The "coarse moduli" viewpoint is relatively imprecise, in comparison with the much richer notion of "fine moduli." For this, one defines for each scheme S the set $F(S)$ of objects to be classified over S. In addition, for each morphism of schemes $u\colon S' \to S$ one provides a base-change map $u^*\colon F(S) \to F(S')$. This gives a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$. One says that M is a fine modular scheme for F if M represents the functor F. This means that M is given along with a "universal object" $\xi \in F(M)$ such that, for each scheme S and each object $\alpha \in F(S)$, there exists a unique morphism $u\colon S \to M$ such that $\alpha = u^*(\xi)$. One then gets an isomorphism of functors $F \approx \mathrm{Hom}(\cdot, M)$.

Grothendieck revealed the utility and flexibility of working with representable functors. In particular, one can give necessary and sufficient conditions on F for M to be proper or smooth. Unfortunately, the functors encountered in classifying abelian varieties are not necessarily representable. Specifically, fix a positive integer g, and define $F(S)$ to be the set of isomorphism classes of principally polarized abelian schemes of dimension g over S. Then F is not representable; the difficulty arises from the fact that principally polarized abelian schemes have nontrivial automorphisms. For example, take $g = 1$. If S is a scheme in which 6 is invertible, the elliptic curves defined by the cubic equations $Y^2Z = X^3 + XZ^2$ and $uY^2Z = X^3 + XZ^2$, with u an invertible function on S, have the same j-invariant but are isomorphic only if u is a square. This circumstance can be traced to the nontrivial involution "multiplication by -1" on elliptic curves. Supplementary difficulties appear near the values $j = 0$ and $j = 1728$; these are due to "extra" automorphisms at these points. Nevertheless, the functor F defined above is "not far" from being representable. It is interesting to recall the various ways in which its failure to be representable has been circumvented.

Let A be an abelian scheme over S, furnished with a principal polarization θ. Let $n \geq 3$ be an integer that is invertible on S. Let ${}_nA$ be the kernel of multiplication by n on A. Serre remarked that every automorphism of (A, θ) that acts trivially on ${}_nA$ is the identity. One can then define over $\mathbf{Z}[1/n]$ the functor F_n that classifies principally polarized g-dimensional abelian schemes (A, θ), furnished with a trivialization of ${}_nA$, i.e., with an isomorphism $\tau\colon (\mathbf{Z}/n\mathbf{Z})^{2g} \approx {}_nA$. This latter functor is representable, by a scheme we can call M_n. The forgetful operation $(A, \theta, \tau) \mapsto (A, \theta)$ leads to a finite morphism $M_n \to M$ over $\mathrm{Spec}(\mathbf{Z}[1/n])$ that is ramified exactly at those points of M that correspond to abelian varieties with exceptional automorphisms.

A more local approach is to start with a polarized abelian variety (A_0, θ_0), defined for instance over a sufficiently large finite extension k of its prime field. The universal deformation of (A_0, θ_0) is a polarized abelian scheme (A, θ) over the spectrum S of a complete local ring with residue field k. The action of the finite group $G = \mathrm{Aut}(A_0, \theta_0)$ lifts to an action of G on (A, θ); further, there exists an action of G on S such that the projection $A \to S$ is G-equivariant. The quotient of S by G is, up to an étale extension, the completion of the coarse moduli scheme M at the point of M that corresponds to (A_0, θ_0). The local obstructions to the representability of F can be read off from the action of G on S.

Inspired by the concept of Grothendieck topologies, Mumford introduced the crucial notion of *stack* in 1963 (see [9]). Although the functor F is not representable, it can be covered by étale families $(A_i, \theta_i) \to S_i$ which, at each closed point of S_i, are algebraizations of universal deformations. Given two such families $(A_i, \theta_i) \to S_i$ $(i = 1, 2)$, one can compare them by introducing the functor "Isom" of isomorphisms between these two objects. This functor is representable by a scheme $S_{1,2}$ that lies above $S_1 \times S_2$ in such a way that the two projections $S_{1,2} \to S_i$ $(i = 1, 2)$ are étale. The collection of the S_i and the $S_{i,j}$ give a description of F that is as flexible and workable as that given by étale charts and coordinate changes in the case of a representable functor. This presentation marked the beginning of the theory of algebraic stacks, which permits a faithful description of moduli problems without any loss of information or need for auxiliary rigidification.

It is impossible to speak of representability without discussing the work of Artin. Artin's marvelous approximation theorem for solutions of algebraic equations over

Hensel rings [1], obtained at the end of the 1960's, led him to introduce the category of *algebraic spaces*, which appear as quotients of schemes by étale equivalence relations. It turns out that the various necessary conditions for representability in the category of schemes, discovered by Grothendieck, remain necessary conditions in the category of algebraic spaces. Further, these conditions, taken together, are *sufficient* for representability in this latter category. This beautiful fact provided a response of unanticipated simplicity to representability questions with which algebraic geometers had struggled for over ten years.

Let us return to the moduli problems, or rather to the algebraic stacks over $\mathbf{Z}$, of genus g curves and principally polarized abelian varieties. These stacks are not proper over $\mathbf{Z}$, and the problem is to compactify them by defining suitable proper stacks over $\mathbf{Z}$ with reasonable geometric descriptions.

This work was carried out for curves by Deligne and Mumford in 1969 [5]. In this case, it "suffices" to compactify families of smooth curves of genus g by adjoining families of semistable curves of genus g whose only singularities are ordinary double points and which are minimal in a simple combinatorial sense. One obtains a proper stack over $\mathbf{Z}$ that furnishes a canonical compactification of the stack of smooth curves. Because of the valuative criterion for properness, the properness of the compactified stack comes down to the following concrete fact, which is quite remarkable: If R is a complete valuation ring, with fraction field K, and if X_K is a smooth proper curve over K, of genus $g \geq 2$, then after a possible finite extension of R, X_K prolongs to a proper semistable curve over R, in an essentially unique way. Although geometers had long been familiar with the singular specializations of smooth curves, they had never suspected that such a simple fact could be true.

When R has residue characteristic zero, the statement about X_K can be obtained by starting with a regular model with normal crossings and then eliminating the multiplicities of the components of the special fiber by replacing R by a finite extension. Historically, however, things worked differently. In the fall of 1964, Grothendieck studied the monodromy of X_K over R via the theory of vanishing cycles that he had just elaborated. He proved that after a possible replacement of R by a finite extension, the specialization of the Jacobian of X_K becomes an extension of an abelian variety by a torus (see [6, Exposé I, §3]). Roughly at the same time, Mumford obtained a similar result using theta functions, at least in residue characteristic different from 2 (cf. [4]). The translation of this result to the reduction of curves came only later. Nowadays we have a few different proofs of the semistable reduction theorem for curves; none, though, is truly elementary in the case where the residue characteristic of R is positive.

Let us turn now to the compactification of the stack M of principally polarized abelian varieties of dimension g. In contrast with the case of curves, there seems to be no natural geometric compactification of M.

If one adopts the viewpoint of group schemes, it is quite clear what sort of degenerations of abelian schemes we should allow at infinity: one should consider *semiabelian* schemes, i.e., smooth group schemes of dimension g whose fibers are extensions of abelian schemes by tori. This means that the objects that we wish to parametrize are no longer themselves proper, and this is a source of considerable technical difficulties.

How can we classify such objects in the neighborhood of a degenerate fiber? Tate was the first to study the case of an elliptic curve E_K, defined over the fraction field of a discrete valuation ring R as above. Assuming that the special fiber of E_K

is a multiplicative group, Tate showed that E_K can be constructed as a quotient of the multiplicative group over K by a group of periods. This multiplicative uniformization is analytic, rather than algebraic, and must be interpreted in the context of rigid analytic spaces, which Tate discovered at this time.

To understand the resulting compactification of the modular variety Spec $\mathbf{Z}[j]$, we must adapt Tate's construction to a "universal" situation where R is replaced by the power series ring $\mathbf{Z}[[q]]$. One can "divide" the multiplicative group $\mathbf{G}_{\mathrm{m}}$ by the powers of q, thereby obtaining a semiabelian scheme that degenerates to the multiplicative group when $q = 0$ and, away from 0, is the elliptic curve with j-invariant $1/q + 744 + 196884q + \cdots$ (see [10]). We obtain in this way a formal chart in a neighborhood of infinity of the compactification of the moduli of elliptic curves. This formal chart can be made algebraic in the following sense: one can find a semiabelian scheme defined over a neighborhood of $j = \infty$ in the projective line over $\mathbf{Z}$, which becomes Tate's degenerate elliptic curve after completion at ∞. This is roughly the idea that Chai and Faltings generalize in higher dimension.

The first step in the process is to describe carefully the local structure of semiabelian schemes in the neighborhood of a degenerate fiber. If A is a semiabelian scheme over S, there is a closed subset S_0 of S over which the toric part of each fiber of A has maximal dimension. Assume that S is affine normal, integral with fraction field K, and complete along S_0. Then A can be partially uniformized in the Tate sense. More precisely, there exists over S a canonical extension E of an abelian scheme B by a torus T, such that A is the quotient of E by a period lattice Λ. The polarization on A gives supplementary structure; in the simplest case where $B = 0$, this is a K-valued quadratic form on the character group of T that satisfies a positivity condition relative to the ideal defining S_0.

To realize A as a quotient of E by Λ, it is apparently necessary to move beyond the techniques of rigid geometry. The authors adopt the method given by Mumford in the case $B = 0$ (in an article that is reproduced as an appendix to the book); this method leads to an economical construction of A that depends on theorems of algebraization of Grothendieck's formal schemes. The uniformization of A provides control of the parameters that describe A in the neighborhood of its degeneration along S_0: these are the parameters of B, the class of the extension E, and the group of periods Λ.

The next step consists in constructing a system of locally universal charts in a neighborhood of infinity. This step depends on the theory of toroidal varieties that was developed in the early 1970s and used by Mumford and his students to obtain compactifications of symmetric domains over $\mathbf{C}$ [2]. The construction depends on a supplementary choice: a simplicial decomposition of the cone of positive $\mathbf{Q}$-valued quadratic forms on $\mathbf{Z}^g$ that is invariant under the action of $\mathrm{GL}(g, \mathbf{Z})$ and finite modulo this action. (The existence of such a decomposition may be viewed as a vast elaboration of Minkowski reduction theory.) To each cell of the decomposition corresponds a certain formal chart that is furnished with a semiabelian scheme that is universal for the periods parameterized by the chart. The authors use Artin's approximation theorem to approximate these formal charts by algebraic charts, which become the étale charts for the compactified stack. This use of the Artin approximation theorem is certainly one of the key points that had blocked the (few) previous workers in the subject.

Although the compactified stack depends on the choice of a simplicial decomposition, any two decompositions can be refined simultaneously by a third. As a result, one ends up with a projective system of compactifications. The stacks that are constructed are smooth (because the polarizations are principal), and the locus at infinity is a divisor with normal crossings (relative to $\mathbf{Z}$). In particular, the structures obtained are "uniform" with respect to the characteristic.

Over the stacks constructed by the authors, there is an invertible sheaf ω: the dual of the highest exterior power of the Lie algebra of the universal semiabelian scheme. The authors show that a positive power of ω is generated by its global sections and thereby obtain by contraction (or blowing down) the sought-after canonical minimal compactification over $\mathbf{Z}$. The latter chapters of the book contain complements that concern modular forms, p-divisible groups, and heights.

This book of over three hundred pages makes no concession to the reader and is not easy to penetrate. It was imperative that a book on this fundamental subject be available quickly, and the mathematical community is deeply indebted to the authors for completing their project in a timely manner. Thanks are also due to the authors and to Springer-Verlag for publishing the book at a reasonable price.

References

1. M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 23–58.
2. A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups Volume IV, Math. Sci. Press, 1975.
3. H. Cartan, *Fonctions automorphes*, Séminaire ENS 1957/58, Benjamin, New York, 1967.
4. C.-L. Chai, *Compactification of Seigel moduli schemes*, London Math. Soc. Lecture Note Ser., vol. 107, Cambridge Univ. Press, Cambridge, 1985.
5. P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 75–109.
6. A. Grothendieck, *Groupes de monodromie en géometrie algébrique*, séminaire de géométrie algébrique du bois-Marie 1967–1969, SGA7 I, Lecture Notes in Math., vol. 288, Springer, New York, 1972.
7. D. Mumford, *Geometric invariant theory*, Ergeb. Math. Grenzgeb. (2), vol. 34, Springer-Verlag, Berlin, 1965.
8. ———, *On equations defining abelian varieties* I, Invent. Math. **1** (1966), 287–354; II, III, Ibid. **3** (1967), 75–135 et 215–244.
9. ———, *Picard groups of moduli problems*, Arithmetical Algebraic Geometry, Proceedings of a Conference at Purdue 1963 (O. F. G. Schilling, ed.), Harper's Series in Modern Mathematics, Harper & Row, New York, 1965.
10. P. Roquette, *Analytic theory of elliptic functions over local fields*, Vandenhoeck & Ruprecht, Göttingen, 1970.

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