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BOOK REVIEW

Discrete subgroups of semisimple Lie groups, by G. A. Margulis. Springer-Verlag, New York, 1991, 387 pp., $85.00. ISBN 3-540-17179-X

The subject of this remarkable book is, as its title indicates, discrete subgroups of semisimple Lie groups. Before describing these objects, it will be useful to recall some basic ideas about Lie groups themselves.

Lie groups arise in a wide variety of situations in geometry and algebra. Being by definition those groups that admit a compatible manifold structure, they arise in geometry as, roughly speaking, the finite-dimensional transformation groups of manifolds or, somewhat more precisely, as the transformation groups of manifolds that can be given locally by finitely many real parameters. While the full diffeomorphism group of a manifold is too large to be finite dimensional with respect to natural topologies, there are many situations where one encounters subgroups that are finite-dimensional Lie groups. One of the most important such geometric situations is that of the isometry group of a Riemannian manifold. In this case it is a classical result of Myers and Steenrod that the isometry group is always a Lie group. The same is true for the symmetry group of certain other classes of geometric structures, e.g., pseudo-Riemannian manifolds, and conformal structures in dimensions at least 3. While there are many natural geometric structures for which the full symmetry group is not necessarily finite dimensional (e.g., volume forms, symplectic structures, complex structures) it is of interest in these cases to understand the finite-dimensional symmetry groups and to understand conditions under which the full symmetry group will be finite dimensional. In an algebraic setting Lie groups arise in a similar manner. The general linear group of a real or complex finite-dimensional vector space is a Lie group. (Of course one can consider this as simply a further example of the symmetry group of a structure on a manifold, namely, a vector space structure.) Closed subgroups will also be Lie groups and, in particular, a subgroup that is the stablizer of any one of the natural objects associated to a vector space will be a Lie group. For example, this is the case for the stabilizer of a tensor, a subspace, a flag, etc. While it is not true that every Lie group (even a connected one) is isomorphic to a linear group (i.e., a subgroup of some general linear group), it is a striking fact that every Lie group is actually locally isomorphic to a linear group.

When one begins to classify Lie groups in terms of their algebraic structure, one is led naturally to the class of semisimple Lie groups, which can be described as those that are locally products of simple Lie groups, and where a simple Lie group is one of dimension at least 2 and with no nontrivial connected normal subgroup. The basic examples of such groups are $\mathrm{SL}(V)$ (the special linear group), and the subgroups leaving invariant a nondegenerate form that is either bilinear and symmetric, Hermitian, or symplectic. While this is one basic manner in which these groups appear algebraically, they appear in a very striking way in geometry; namely, among the Riemannian manifolds there is the very natural class of symmetric spaces. These are the simply connected complete Riemannian manifolds with the property that the sectional curvature is invariant under parallel translation. (The standard definition allows a somewhat more general class, including some nonsimply connected spaces. However, this definition will suffice for the purposes of this review.) This obviously includes the complete simply connected spaces of constant curvature but contains many other Riemannian manifolds as well. It is a basic fact that the isometry group of a symmetric space [which one knows (although we are not being historically faithful here) is a Lie group by the Myers-Steenrod theorem] is in fact a semisimple Lie group (acting transitively) as long as there is not a direct factor that is a Euclidean space and that conversely every semisimple Lie group arises locally as the (transitive) isometry group of a Riemannian symmetric space with no Euclidean direct factor. Semisimple Lie groups are not just locally linear in some abstract manner but very concretely via a canonical representation (the adjoint representation on the Lie algebra). This linear realization allows one to understand the structural and representation theoretic properties of these groups in great detail. While these are useful in virtually all situations in which semisimple groups arise, they in particular have natural geometric meaning in terms of the symmetric space associated to the group and, in fact, yield a very fine understanding of the geometric properties of symmetric spaces. This theory can be viewed as a very successful linearization or algebraization of the natural geometric problem of understanding the geometry of symmetric spaces.

Once one considers linear Lie groups one can ask whether or not the group is (real) algebraic, i.e., not only a submanifold of $\mathrm{GL}(V)$ but (the real points of) a subvariety (defined over $\mathbb{R}$). In other words, one asks if the group can be given as the set of zeros of a family of polynomial functions rather than only smooth functions. The advantage of this is, of course, that one can bring to bear the apparatus of commutative algebra and algebraic geometry. Not every Lie group is locally isomorphic to a real algebraic group. For the examples of semisimple groups described above, namely, those preserving a suitable form or the determinant, it is essentially immediate from the definition that these groups are algebraic. Furthermore, from some basic results about Lie algebras it follows easily that for semisimple Lie groups in general the canonical (local) linear realization given by the adjoint representation exhibits the group as being locally isomorphic to an algebraic one. In addition, one can show that this structure of an algebraic group is essentially unique over the complex numbers. Given an algebraic group (say over $\mathbb{R}$), one can ask when it has a k-structure, where k is a subfield. This means that there is a linear realization in which the group is given by the set of zeros of a family of polynomials with coefficients in k. There are examples of Lie groups that are real algebraic groups but which have no $\mathbb{Q}$-structure. However, every semisimple algebraic group admits a $\mathbb{Q}$-structure. This is clear for the above examples (simply by taking the relevant form to have rational coefficients), and, in general, it follows by a careful analysis of the adjoint representation. However, this $\mathbb{Q}$-structure is no longer unique. The study of the k-structures on a semisimple algebraic group where k is an algebraic number field is a basic and difficult problem in the "arithmetic theory" of algebraic groups. We shall see its relevance to discrete subgroups and geometry in a moment.

We now turn to the subject of the book at hand, namely, discrete subgroups. These also occur in a variety of ways, but we shall focus on two salient situations. Suppose we consider locally symmetric spaces, namely, those complete Riemannian manifolds with sectional curvature invariant under parallel translation. Thus, these differ from symmetric spaces only in that we no longer assume simple connectivity. This class now includes all complete (e.g., compact) manifolds of constant curvature. The universal covering space of a locally symmetric space is symmetric, and hence we can view any locally symmetric space M as X/D where X is symmetric and D is the fundamental group of M acting as isometries of the space X. In particular, D is a subgroup of the isometry group of X, which as we have observed as a Lie group, and it is easy to see from this construction that D is in fact discrete. Conversely, given any discrete subgroup D of the isometry group that is torsionfree, we can form the quotient X/D, which will be a locally symmetric space with fundamental group D. (The condition that D be torsionfree can always be achieved by passing to a subgroup of finite index. For the remainder of our discussion, we shall ignore this technicality.) In other words, understanding the discrete subgroups of a semisimple Lie group is tantamount to understanding the fundamental groups of locally symmetric spaces that are covered by the symmetric space corresponding to the Lie group. In many cases of interest, for example, if the semisimple group is a product of noncompact simple groups, then the symmetric space can be shown to be contractible, and hence the fundamental group of the locally symmetric space M carries all the homotopy information of M. From the point of view of geometry, we can view understanding the relation of a discrete subgroup D to the ambient semisimple group G as a version of the question of understanding the relationship between the topology of M (via the fundamental group D) and the local geometry of a locally symmetric space M (incorporated in G via the universal cover of M). Of particular interest is of course the case in which M is compact, which is equivalent to D being a cocompact subgroup of G (i.e., G/D is compact). A more general situation is one in which M is not necessarily compact but has finite volume, which corresponds to G/D being of finite volume with respect to the Haar measure on G. In this case D is called a lattice in G.

The simplest algebraic setting in which discrete subgroups arise is most easily illustrated by considering the subgroups $D = \mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$. Discreteness, of course, follows from the discreteness of $\mathbb{Z}$ in $\mathbb{R}$. In this case it is a classical result that D is a lattice but is not cocompact. For $n = 2$ both of these assertions are easily seen from the standard fundamental domain for $\mathrm{SL}(2, \mathbb{Z})$ in the upper half plane. This example can be generalized and illuminated by the following general construction. Suppose G is a semisimple algebraic group with a $\mathbb{Q}$-structure. Let D be the elements of G that consist of matrices with integer entries. Then D is a lattice subgroup of G that may or may not be cocompact. (This is due to Borel and Harish-Chandra.) The group D is actually not quite uniquely determined by the $\mathbb{Q}$-structure. Namely, if one has two different $\mathbb{Q}$-structures that are isomorphic in the natural sense, then the corresponding discrete subgroups will be commensurable, in the sense that the intersection is of finite index in each of them. Thus, a $\mathbb{Q}$-structure on G determines a commensurability class of lattices in G, and such a discrete subgroup is called an arithmetic subgroup of G. More generally, instead of a $\mathbb{Q}$-structure one can consider a k-structure where k is an algebraic number field, and the subgroup of G consisting of matrices with entries in $\mathcal{O}$, the algebraic integers in k. Since $\mathcal{O}$ is no longer necessarily discrete in $\mathbb{C}$, this will not, in general, yield a discrete subgroup of G. However, in certain readily describable situations this group will actually be discrete, and subgroups commensurable with these groups are also called arithmetic subgroups of G. (In the cases in which the subgroup is not discrete, it will in fact be an arithmetic lattice in a larger group, namely, one that is a product of G with another explicitly describable semisimple group.) Summarizing, we see that $\mathbb{Q}$-structures, and in some cases k-structures, define a class of lattices, namely, the arithmetic ones. By their very construction, these groups arise naturally in a wide variety of number theoretic problems, and the properties of these groups and their relation to G is a basic issue in number theory. From the geometric point of view, one can expect that arithmetic lattices will be more tractable than a general lattice, by virtue of the explicit nature of their construction. This can be nicely illustrated by volume computations, where one can compute the volume of the associated locally symmetric space in terms of the algebraic data defining the arithmetic group and try to make statements about the collection of all such volumes.

While the above discussion hopefully provides some motivation for the study of lattice subgroups (or at least the plausibility that such motivation abounds), we now discuss some of the specific questions about these groups that Margulis considers in his book. These questions are, generally speaking, those fundamental questions to which Margulis himself provided the answer in the 1970s with his dramatic and systematic use of ergodic theoretic and related techniques in resolving issues that had resisted more (at the time) traditional approaches, as well as some background results needed for this development. A general perspective on lattice subgroups is that in many cases, one would expect to see manifestations of properties of the ambient semisimple group reflected in properties of the lattice subgroup. (In light of our discussion above, this is an algebraic version of the geometric idea that one should see the influence of the local geometry of a manifold on the structure and properties of the fundamental group.) For many issues, this manifestation is strongest in the case of semisimple groups of real rank at least 2. This means that there is a locally isomorphic real linear group that contains a two-dimensional subgroup of diagonal matrices. This includes $\mathrm{SL}(n, \mathbb{R})$ for $n > 2$ but excludes $\mathrm{SL}(2, \mathbb{R})$. Similarly, it includes $\mathrm{SO}(p, q)$ for $p + q > 4$ and $p, q > 1$ but excludes the groups $\mathrm{SO}(1, n)$. The deepest results Margulis describes are for lattices in groups with this property. Many of the results, either explicitly or implicitly, center around identifying or at least describing salient features of homomorphisms of a lattice D into some other class of natural groups. For example, Margulis describes his superrigidity theorem, identifying the finite-dimensional linear representations of lattices, and allowing him to establish arithmeticity of a large class of lattices. He studies surjections onto a class of discrete groups, allowing him to show that the normal subgroups of many lattices are either finite or of finite index. He also studies homomorphisms into automorphism groups of trees, allowing him to show that a large class of lattices cannot be written as a nontrivial amalgamated product. The unifying theme in his approach to many of these results is first to reduce them to issues regarding the action of D on natural algebraic homogeneous spaces of G and then to study the relevant properties of this action. The properties that arise and the techniques used to study these actions are those of ergodic theory (i.e., the measure theoretic properties of group actions) and related dynamical techniques that are among the natural techniques for studying actions of noncompact groups in general and discrete groups acting on homogeneous spaces in particular. Margulis also develops the relevant ergodic theoretic machinery in this book.

To give some further flavor of the nature of the results, we briefly discuss as an example Margulis' arithmeticity theorem. Margulis provides a complete discussion of his proof of the assertion that for groups of real rank at least 2, all lattices are arithmetic, i.e., given by the construction outlined above. [Actually, one needs to make a further technical assumption ("irreducibility") to avoid the situation of a product of lattices in a product of rank one groups.] This remarkable theorem, which Margulis proved in 1974, provides in principle an algebraization of the problem of understanding the fundamental groups of locally symmetric spaces (of rank at least 2). As we indicated above, the classical theory of symmetric spaces shows how to describe symmetric spaces algebraically, Margulis's theorem brings the same sort of order to the locally symmetric spaces, at least to the extent that one can effectively describe the relevant k-structures on the ambient semismple groups. (We should remark here that this latter problem is itself highly nontrivial, involves a large amount of arithmetic work, and is not dealt with in this book.) The proof of arithmeticity follows by purely algebraic arguments from enough knowledge of the finite-dimensional linear representation theory of D. The required representation theoretic information is contained in Margulis's superrigidity theorem, which asserts roughly that every representation of D over a local field can be understood in terms of representations that extend continuously to G or in terms of representations with precompact image. It is in proving this result that Margulis uses and develops ergodic theoretic arguments.

The book is very carefully written and, except for some results on algebraic groups, is self-contained. Margulis works in a very general framework, considering not only semisimple Lie groups but products of semisimple Lie groups over varying local fields and discrete subgroups of these groups. The book is both accessible (particularly to those with algebraic background) and, by virtue of its high level of completeness, will serve as an excellent reference as well. In addition to the central results and themes in the book there is a wealth of other information, including new and simpler proofs of many known results. The book will no doubt instantly become a classic in the field.

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