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PIERCING CONVEX SETS

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ABSTRACT. A family of sets has the (p, q) property if among any p members of the family some q have a nonempty intersection. It is shown that for every $p \geq q \geq d+1$ there is a $c = c(p, q, d) < \infty$ such that for every family $\mathcal{F}$ of compact, convex sets in R^d that has the (p, q) property there is a set of at most c points in R^d that intersects each member of $\mathcal{F}$. This extends Helly's Theorem and settles an old problem of Hadwiger and Debrunner.

1. INTRODUCTION

For two integers $p \geq q$, a family of sets $\mathcal{H}$ has the (p, q) *property* if among any p members of the family some q have a nonempty intersection. $\mathcal{H}$ is k*-pierceable* if it can be split into k or fewer subfamilies, each having a nonempty intersection. The *piercing number* of $\mathcal{H}$, denoted by $P(\mathcal{H})$, is the minimum value of k such that $\mathcal{H}$ is k-pierceable. (If no such finite k exists then $P(\mathcal{H}) = \infty$.)

The classical theorem of Helly [14] states that any family of compact convex sets in R^d that satisfies the $(d+1, d+1)$-property is 1-pierceable. Hadwiger and Debrunner considered the more general problem of studying the piercing numbers of families $\mathcal{F}$ of compact, convex sets in R^d that satisfy the (p, q) property. By considering the intersections of hyperplanes in general position in R^d with an appropriate box one easily checks that for $q \leq d$ the piercing number can be infinite, even if $p = q$. Thus we may assume that $p \geq q \geq d+1$.

Let $M(p, q; d)$ denote the maximum possible piercing number (which is possibly infinity) of a family of compact convex sets in R^d with the (p, q)-property. By Helly's Theorem,

$$M(d+1, d+1; d) = 1$$

for all d and, trivially, $M(p, q; d) \geq p-q+1$. Hadwiger and Debrunner [12] proved that for $p \geq q \geq d+1$ satisfying

$$p(d-1) < (q-1)d, \tag{1}$$

this bound is tight, i.e., $M(p, q; d) = p-q+1$. In all other cases, it is not even known if $M(p, q; d)$ is finite, and the question of deciding if this function is finite, raised by Hadwiger and Debrunner in 1957 in [12], remained open. This question, which is usually referred to as the (p, q)-problem, is considered in various survey articles and books, including [13, 5, 8]. The smallest case in which finiteness is unknown, which is pointed out in all the above-mentioned

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articles, is the special case $p = 4$, $q = 3$, $d = 2$. We note, in all the cases where finiteness is known, that in fact $M(p, q; d) = p - q + 1$ and there are examples of Danzer and Grünbaum (cf. [13]) that show $M(4, 3; 2) \geq 3 > 4 - 3 + 1$.

The (p, q)-problem received a considerable amount of attention, and finiteness was proved for various restricted classes of convex sets, including the family of parallelotopes with edges parallel to the coordinate axes in R^d [13, 19, 6], families of homothetes of a convex set [19], and, using a similar approach, families of convex sets with a certain "squareness" property ([9], see also [21]).

Despite these efforts, the problem of deciding if $M(p, q; d)$ is finite remained open for all values of $p \geq q \geq d + 1$ that do not satisfy (1).

The purpose of the present note is to announce the solution of this problem, established in the following

Theorem 1.1. *For every* $p \geq q \geq d + 1$ *there is a* $c = c(p, q, d) < \infty$ *such that* $M(p, q; d) \leq c$; *i.e., for every family* $\mathscr{F}$ *of compact, convex sets in* R^d *that has the* (p, q) *property there is a set of at most* c *points in* R^d *that intersects each member of* $\mathscr{F}$.

The detailed proof will appear in [3]. Here we briefly sketch the main ideas. Three tools are applied: a fractional version of Helly's Theorem that was first proved in [15], Farkas's Lemma (or Linear Programming Duality), and a recent result proved in [1].

It may seem that there are almost no interesting families of compact convex sets in R^d that satisfy the (p, q)-property for some $p \geq q \geq d+1$. A large class of examples can be constructed as follows. Let μ be an arbitrary probability distribution on R^d, and let $\mathscr{F}$ be the family of all compact convex sets F in R^d satisfying $\mu(F) \geq \varepsilon$. Since the sum of the measures of any set of more than d/ε such sets is greater than d, it follows that if p is the smallest integer strictly larger than d/ε then $\mathscr{F}$ has the $(p, d + 1)$ property. It then follows that $P(\mathscr{F}) \leq M(p, d + 1; d + 1)$, i.e., for every probability measure in R^d there is a set X of at most $M(p, d + 1; d + 1)$ points such that any compact convex set in R^d whose measure exceeds ε intersects X.

The following theorem is an immediate consequence of Theorem 1.1.

Theorem 1.2. *Let* $\mathscr{F}$ *be a family of compact convex sets in* R^d, *and suppose that for every subfamily* $\mathscr{F}'$ *of cardinality* x *of* $\mathscr{F}$ *the inequality* $P(\mathscr{F}') < \lceil x/d \rceil$ *holds; i.e.,* $\mathscr{F}'$ *can be pierced by less than* x/d *points. Then* $P(\mathscr{F}) \leq M(x, d + 1; d + 1)$.

Observe that in order to deduce a finite upper bound for the piercing number of $\mathscr{F}$, the assumption that $\mathscr{P}(\mathscr{F}') < \lceil x/d \rceil$ cannot be replaced by $P(\mathscr{F}') \leq \lceil x/d \rceil$ as shown by an infinite family of hyperplanes in general position (intersected with an appropriate box), whose piercing number is infinite.

2. A sketch of the proofs

Since we do not try to optimize the constants here, and since obviously $M(p, q; d) \leq M(p, d + 1; d)$ for all $p \geq q \geq d + 1$, it suffices to prove an upper bound for $M(p, d + 1; d)$. Another simple observation is that by compactness we can restrict our attention to finite families of convex sets.

Let $\mathscr{F}$ be a family of n convex sets in R^d, and suppose that $\mathscr{F}$ has the $(p, d + 1)$ property. Our objective is to find an upper bound for the piercing number $P(\mathcal{F})$ of $\mathcal{F}$, where the bound depends only on p and d. It is convenient to describe the ideas in three subsections.

2.1. **A fractional version of Helly's Theorem.** Katchalski and Liu [15] proved the following result, which can be viewed as a fractional version of Helly's Theorem.

Theorem 2.1 [15]. *For every $0 < \alpha \leq 1$ and every d there is a $\delta = \delta(\alpha, d) > 0$ such that for every $n \geq d+1$, every family of n convex sets in R^d that contains at least $\alpha\binom{n}{d+1}$ intersecting subfamilies of cardinality $d+1$ contains an intersecting subfamily of at least δn of the sets.*

Notice that Helly's Theorem is equivalent to the statement that in the above theorem $\delta(1, d) = 1$.

A sharp quantitative version of this theorem was proved by Kalai [16] and, independently, by Eckhoff [7]. See also [2] for a very short proof. All these proofs rely on Wegner's Theorem [20] that asserts that the nerve of a family of convex sets in R^d is d-collapsible.

Theorem 2.1, together with a simple probabilistic argument, can be applied to prove

Lemma 2.2. *For every $p \geq d+1$ there is a positive constant $\beta = \beta(p, d)$ with the following property. Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d that has the $(p, d+1)$ property. Let a_i be nonnegative integers, define $m = \sum_{i=1}^n a_i$, and let $\mathcal{G}$ be the family of cardinality m consisting of a_i copies of A_i for $1 \leq i \leq n$. Then there is a point x in R^d that belongs to at least βm members of $\mathcal{G}$.*

2.2. **Farkas's Lemma and a lemma on hypergraphs.** The following is a known variant of the well-known lemma of Farkas (cf. [17, p. 90]).

Lemma 2.3. *Let A be a real matrix and b a real (column) vector. Then the system $Ax \leq b$ has a solution $x \geq 0$ if and only if for every (row) vector $y \geq 0$ that satisfies $yA \geq 0$ the inequality $yb \geq 0$ holds.*

This lemma can be used to prove the following.

Corollary 2.4. *Let $H = (V, E)$ be a hypergraph and let $0 \leq \gamma \leq 1$ be a real. Then the following two conditions are equivalent:*

(i) *There exists a weight function $f\colon V \mapsto R^+$ satisfying $\sum_{v\in V} f(v) = 1$ and $\sum_{v\in e} f(v) \geq \gamma$ for all $e \in E$.*
(ii) *For every function $g\colon E \mapsto R^+$ there is a vertex $v \in V$ such that $\sum_{e;v\in e} g(e) \geq \gamma \sum_{e\in E} g(e)$.*

By the last corollary and Lemma 2.2 one can prove the following result.

Corollary 2.5. *Suppose $p \geq d+1$ and let $\beta = \beta(p, d)$ be the constant from Lemma 2.2. Then for every family $\mathcal{F} = \{A_1, \ldots, A_n\}$ of n convex sets in R^d with the $(p, d+1)$ property there is a finite (multi-)set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta|Y|$ for all $1 \leq i \leq n$.*

2.3. **Weak ε-nets for convex sets.** The following result is proved in [1].

Theorem 2.6 [1]. *For every real $0 < \varepsilon < 1$ and every integer d there exists a constant $b = b(\varepsilon, d)$ such that the following holds:*

For every m and every multiset Y of m points in R^d, there is a subset X of at most b points in R^d such that the convex hull of any subset of εm members of Y contains at least one point of X.

Several arguments that supply various upper bounds for $b(\varepsilon, d)$ are given in [1]. The simplest one is based on a result of Bárány [4] whose proof is based on a deep result of Tverberg [8].

Theorem 1.1 follows from the above results quite easily. Let $\mathscr{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d with the $(p, d+1)$ property, where $p \geq d+1$. By Corollary 2.5 there is a finite (multi-)set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta|Y|$ for all $1 \leq i \leq n$, where $\beta = \beta(p, d)$ is as in Lemma 2.2. By Theorem 2.6 there is a set X of at most $b(\beta, d)$ points in R^d such that the convex hull of any set of $\beta|Y|$ members of Y contains at least one point of X. Since each member of $\mathscr{F}$ contains at least $\beta|Y|$ points in Y, it must contain at least one point of X. Therefore, $P(\mathscr{F}) \leq |X| \leq b(\beta(p, d), d)$, completing the proof.

The detailed proofs of the above lemmas and corollaries, as well as some methods to improve the estimates for the numbers $M(p, q; d)$, will appear in [3].

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