

NEW TYPES OF SOLITON SOLUTIONS

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ABSTRACT. We announce a detailed investigation of limits of N -soliton solutions of the Korteweg-deVries (KdV) equation as N tends to infinity. Our main results provide new classes of KdV-solutions including in particular new types of soliton-like (reflectionless) solutions. As a byproduct we solve an inverse spectral problem for one-dimensional Schrödinger operators and explicitly construct smooth and real-valued potentials that yield a purely absolutely continuous spectrum on the nonnegative real axis and give rise to an eigenvalue spectrum that includes any prescribed countable and bounded subset of the negative real axis.

INTRODUCTION

In this note we announce the construction of new types of soliton-like solutions of the Korteweg-de Vries (KdV)-equation. More precisely, we offer a solution to the following problem:

Construct new classes of KdV-solutions by taking limits of N -soliton solutions as $N \rightarrow \infty$.

As it turns out, our solution to this problem is intimately connected with a solution to the following inverse spectral problem in connection with one-dimensional Schrödinger operators $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R})$:

Given any bounded and countable subset $\{-\kappa_j^2\}_{j \in \mathbb{N}}$ of $(-\infty, 0)$, construct a (smooth and real-valued) potential V such that $H = -d^2/dx^2 + V$ has a purely absolutely continuous spectrum equal to $[0, \infty)$ and the set of eigenvalues of H includes the prescribed set $\{-\kappa_j^2\}_{j \in \mathbb{N}}$.

In addition, we also construct a new class of reflectionless KdV-solutions in which the underlying Schrödinger operator has infinitely many negative eigenvalues accumulating at zero.

Although we present our results exclusively in the KdV-context, it will become clear later on that our methods are not confined to KdV-type equations but are widely applicable in the field of integrable systems.

Received by the editors November 19, 1991

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35Q51, 35Q53; Secondary 58F07

The first two authors were partially supported by the BiBoS-Research Center at the Faculty of Physics, University of Bielefeld and the Department of Mathematics, University of Bochum, FRG. The first author was also partially supported by the Norwegian Research Council for Science and the Humanities during a stay at the Division of Mathematical Sciences of the University of Trondheim, Norway. The second author was also partially supported by CNRS-Marseille, the University of Provence I, Marseille, France, and the German-Polish scientific collaboration program

Before formulating our results in detail we briefly review some background material. The celebrated N -soliton solutions $V_N(t, x)$ of the KdV-equation

$$(1) \quad \text{KdV}(V) = V_t - 6VV_x + V_{xxx} = 0$$

described, e.g., in [1, 2, 4]

$$(2) \quad \begin{aligned} V_N(t, x) &= -2\partial_x^2 \ln \det[1_N + C_N(t, x)], \quad (t, x) \in \mathbb{R}^2, \\ C_N(t, x) &= \left[\frac{c_j c_\ell}{\kappa_j + \kappa_\ell} e^{4(\kappa_j^3 + \kappa_\ell^3)t - (\kappa_j + \kappa_\ell)x} \right]_{j, \ell=1}^N, \\ &c_j > 0, \quad \kappa_j > 0, \quad 1 \leq j \leq N, \quad N \in \mathbb{N} \end{aligned}$$

are well known to be isospectral and reflectionless potentials $V_N(t, x)$ in connection with the one-dimensional Schrödinger operator

$$(3) \quad H_N(t) = -\frac{d^2}{dx^2} + V_N(t, \cdot)$$

in $L^2(\mathbb{R})$. In particular, the spectrum $\sigma(H_N(t))$ of $H_N(t)$ is independent of t and is given by

$$(4) \quad \sigma(H_N(t)) = \{-\kappa_j^2\}_{j=1}^N \cup [0, \infty)$$

with purely absolutely continuous spectrum $[0, \infty)$. Hence $H_N(t)$ are isospectral deformations of $H_N(0)$, which is clear from the Lax formalism connecting (1) and (3). The reflectionless property of V_N manifests itself in the (t -independent) scattering matrix $S_N(k)$ in \mathbb{C}^2 associated with the pair $(H_N(t), H_0)$

$$S_N(k) = \begin{pmatrix} T_N(k) & 0 \\ 0 & T_N(k) \end{pmatrix}, \quad T_N(k) = \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad k \in \mathbb{C} \setminus \{i\kappa_j\}_{j=1}^N,$$

where $H_0 = -d^2/dx^2$ and $z = k^2$ is the spectral parameter corresponding to H_0 . Here $T_N(k)$ denotes the transmission coefficient and the vanishing of the off-diagonal terms in $S_N(k)$ exhibits reflection coefficients identical to zero at all energies. In more intuitive terms this remarkable and highly exceptional behavior can be described as follows: If one views V as representing an “obstacle” for an incoming “signal” (wave, etc.) then the outgoing signal generically consists of two parts, a transmitted and a reflected one. It is in the exceptional case of N -soliton potentials V_N such as (2) that the reflected part of the outgoing signal is entirely missing and hence the obstacle appears to be completely transparent independently of the wavelength of the incoming signal.

Incidentally, (4) offers a solution to the following inverse spectral problem: Given the finite set $\{-\kappa_j^2\}_{j=1}^N \subset (-\infty, 0)$, construct (smooth and real-valued) potentials V_N such that $H_N = -d^2/dx^2 + V_N$ has a purely absolutely continuous spectrum equal to $[0, \infty)$ and precisely the eigenvalues $\{-\kappa_j^2\}_{j=1}^N$.

A natural generalization of this fact would be to ask whether one can choose a sequence $\{c_j > 0\}_{j \in \mathbb{N}}$ such that for an arbitrarily prescribed bounded and countable set $\{-\kappa_j^2\}_{j \in \mathbb{N}} \subset (-\infty, 0)$, $V_N(t, x)$ converge to a smooth KdV-solution $V_\infty(t, x)$ as $N \rightarrow \infty$ with the associated Schrödinger operator $H_\infty(t) = -d^2/dx^2 + V_\infty(t, \cdot)$ having the purely absolutely continuous spectrum $[0, \infty)$ and containing the set $\{-\kappa_j^2\}_{j \in \mathbb{N}}$ in its point spectrum.

The main goal of this note is to present an affirmative answer to this question.

MAIN RESULTS

Theorem 1. Assume $\{\kappa_j > 0\}_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$, $\kappa_j \neq \kappa_\ell$ for $j \neq \ell$, and choose $\{c_j > 0\}_{j \in \mathbb{N}}$ such that $\{c_j^2/\kappa_j\}_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then V_N converges pointwise to some $V_\infty \in C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ as $N \rightarrow \infty$ and

(i) $\lim_{x \rightarrow +\infty} V_\infty(t, x) = 0$ and

$$(5) \quad \lim_{N \rightarrow \infty} \sup_{(t, x) \in K} |\partial_t^m \partial_x^n V_N(t, x) - \partial_t^m \partial_x^n V_\infty(t, x)| = 0, \quad m, n \in \mathbb{N}_0,$$

for any compact subset $K \subset \mathbb{R}^2$. Moreover,

$$(6) \quad KdV(V_\infty) = 0.$$

(ii) Denoting $H_\infty(t) = -d^2/dx^2 + V_\infty(t, \cdot)$ we have

$$(7) \quad \sigma_{\text{ess}}(H_\infty(t)) = \{-\kappa_j^2\}'_{j \in \mathbb{N}} \cup [0, \infty),$$

$$(8) \quad \sigma_{\text{ac}}(H_\infty(t)) = [0, \infty),$$

$$(9) \quad [\sigma_p(H_\infty(t)) \cup \sigma_{\text{sc}}(H_\infty(t))] \cap (0, \infty) = \emptyset,$$

$$(10) \quad \{-\kappa_j^2\}_{j \in \mathbb{N}} \subseteq \sigma_p(H_\infty(t)) \subseteq \overline{\{-\kappa_j^2\}_{j \in \mathbb{N}}}.$$

The spectral multiplicity of $H_\infty(t)$ on $(0, \infty)$ equals two while $\sigma_p(H_\infty(t))$ is simple. In addition, if $\{\kappa_j\}_{j \in \mathbb{N}}$ is a discrete subset of $(0, \infty)$ (i.e., if 0 is its only limit point) then

$$(11) \quad \sigma_{\text{sc}}(H_\infty(t)) = \emptyset,$$

$$(12) \quad \sigma(H_\infty(t)) \cap (-\infty, 0) = \sigma_d(H_\infty(t)) = \{-\kappa_j^2\}_{j \in \mathbb{N}}.$$

More generally, if $\{-\kappa_j^2\}'_{j \in \mathbb{N}}$ is countable then (11) holds.

Here \bar{A} denotes the closure of $A \subset \mathbb{R}$, A' is the derived set of A (i.e., the set of accumulation points of A), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\sigma_{\text{ess}}(\cdot)$, $\sigma_{\text{ac}}(\cdot)$, $\sigma_{\text{sc}}(\cdot)$, $\sigma_d(\cdot)$, and $\sigma_p(\cdot)$ denote the essential, absolutely continuous, singularly continuous, discrete, and point spectrum (the set of eigenvalues) respectively.

Under stronger assumptions on $\{\kappa_j\}_{j \in \mathbb{N}}$ we obtain

Theorem 2. Assume $\{\kappa_j > 0\}_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$, $\kappa_j \neq \kappa_\ell$ for $j \neq \ell$, and choose $\{c_j > 0\}_{j \in \mathbb{N}}$ such that $\{c_j^2/\kappa_j\}_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then in addition to (5) and (6) we have

(i)

$$(13) \quad \lim_{N \rightarrow \infty} \|\partial_t^m \partial_x^n V_N(t, \cdot) - \partial_t^m \partial_x^n V_\infty(t, \cdot)\|_p = 0, \quad m, n \in \mathbb{N}_0, \quad 1 \leq p \leq \infty.$$

(ii)

$$\begin{aligned} \sigma_{\text{ess}}(H_\infty(t)) &= \sigma_{\text{ac}}(H_\infty(t)) = [0, \infty), \\ \sigma_p(H_\infty(t)) \cap (0, \infty) &= \sigma_{\text{sc}}(H_\infty(t)) = \emptyset, \\ \sigma_d(H_\infty(t)) &= \{-\kappa_j^2\}_{j \in \mathbb{N}}. \end{aligned}$$

(iii) The (t -independent) scattering matrix $S_\infty(k)$ in \mathbb{C}^2 associated with the pair $(H_\infty(t), H_0)$ is reflectionless and given by

$$S_\infty(k) = \begin{pmatrix} T_\infty(k) & 0 \\ 0 & T_\infty(k) \end{pmatrix},$$

$$T_\infty(k) = \prod_{j=1}^{\infty} \frac{k + i\kappa_j}{k - i\kappa_j}, \quad k \in \mathbb{C} \setminus \{i\kappa_j\}_{j \in \mathbb{N}} \cup \{0\}.$$

While Theorem 1 solves the two problems stated in the introduction, Theorem 2 constructs a new class of reflectionless potentials in connection with one-dimensional Schrödinger operators involving an infinite negative point spectrum accumulating at zero. Moreover, suppose $V \in C^\infty(\mathbb{R}^2)$ to be real-valued with $\partial_x^m V(t, \cdot) \in L^1(\mathbb{R})$, $m \in \mathbb{N}$, and either $V(t, \cdot) \in L^1(\mathbb{R}; (1+|x|^\varepsilon) dx)$ for some $\varepsilon > 0$ or that $T(k)$, $k \in \mathbb{R} \setminus \{0\}$, the transmission coefficient associated with the pair $(H(t) = -d^2/dx^2 + V(t, \cdot), H_0)$, has an analytic continuation into $\{\mathbb{C}_+ \setminus \{i\kappa_j\}_{-\kappa_j^2 \in \sigma_d(H(0))}\} \cup \{k \in \mathbb{C} \setminus \{0\} \mid |k| < \eta, \text{Im } k \leq 0\}$ for some $\eta > 0$ ($\mathbb{C}_+ = \{k \in \mathbb{C} \mid \text{Im } k > 0\}$). Introducing the KdV-invariants $\chi_n \in C^\infty(\mathbb{R}^2)$ by

$$\chi_1 = V, \quad \chi_2 = -V_x, \quad \chi_{n+1} = -\partial_x \chi_n - \sum_{m=1}^{n-1} \chi_{n-m} \chi_m, \quad n \geq 2,$$

an extension of the results in [5, 13] yields the conservation laws (trace relations)

$$(14) \quad - \int_{\mathbb{R}} dx \chi_{2n+1}(t, x) = \frac{2^{2(n+1)}}{2n+1} \sum_{-\kappa_j^2 \in \sigma_d(H)} \kappa_j^{2n+1} \\ + (-1)^n 2^{2(n+1)} \frac{1}{\pi} \int_0^\infty dk k^{2n} \ln |T(k)|, \quad n \in \mathbb{N}_0$$

assuming $\sum_{-\kappa_j^2 \in \sigma_d(H)} \kappa_j < \infty$ (see [6] for details). Since $|T(k)| \leq 1$ for $k > 0$ by the unitarity of the associated scattering matrix, this yields the bounds

$$(15) \quad - \int_{\mathbb{R}} dx \chi_{4m+1}(t, x) \leq \frac{2^{2(2m+1)}}{4m+1} \sum_{-\kappa_j^2 \in \sigma_d(H)} \kappa_j^{4m+1}, \quad m \in \mathbb{N}_0,$$

$$(16) \quad - \int_{\mathbb{R}} dx \chi_{4m+3}(t, x) \geq \frac{2^{2(2m+2)}}{4m+3} \sum_{-\kappa_j^2 \in \sigma_d(H)} \kappa_j^{4m+3}, \quad m \in \mathbb{N}_0.$$

For $m = 0$ the bound (15) can be found in [12]. In the case where $\partial_x^m V(t, \cdot) \in L^1(\mathbb{R}; (1+|x|) dx)$, $m \in \mathbb{N}_0$, and hence $\sigma_d(H(t))$ is finite, (15) and (16) are discussed, e.g., in [8, 10]. By (14), the bounds (15) and (16) saturate iff $|T(k)| = 1$ for $k > 0$, i.e., iff V is reflectionless. Consequently, the bounds (15) and (16) saturate if V equals the N -soliton solutions V_N in (2) and, in particular, if V is an element of our new class of reflectionless KdV-solutions V_∞ described in Theorem 2.

SKETCH OF PROOFS

The hypotheses in Theorem 1 guarantee that $C_N(t, x)$ (viewed as an operator in $\ell^2(\mathbb{N})$) converges for any fixed $(t, x) \in \mathbb{R}^2$ in trace norm to some trace class operator $C_\infty(t, x) \in \mathcal{B}_1(\ell^2(\mathbb{N}))$ as $N \rightarrow \infty$ and hence

$$\lim_{N \rightarrow \infty} V_N(t, x) = V_\infty(t, x) = -2\partial_x^2 \ln \det_1[1 + C_\infty(t, x)],$$

where $\det_1(\cdot)$ denotes the corresponding Fredholm determinant. A crucial identity in proving (5) and (6) is

$$(17) \quad V_\infty(t, x) = -4 \sum_{j=1}^{\infty} \kappa_j \psi_{\infty, j}(t, x)^2,$$

where $\{\psi_{\infty, j}(t, x)\}_{j \in \mathbb{N}}$ turn out to be the eigenfunctions of $H_\infty(t)$ corresponding to the eigenvalues $\{-\kappa_j^2\}_{j \in \mathbb{N}}$, determined by

$$(18) \quad \begin{aligned} \Psi_\infty(t, x) &= [1 + C_\infty(t, x)]^{-1} \Psi_\infty^0(t, x), \\ \Psi_\infty^0(t, x) &= \{c_j e^{-\kappa_j x}\}_{j \in \mathbb{N}}^T, \\ \Psi_\infty(t, x) &= \{\psi_{\infty, j}(t, x)\}_{j \in \mathbb{N}}^T \end{aligned}$$

in $\ell^2(\mathbb{N})$. Equations (17) and (18) are well known in the context of V_N and can be obtained by pointwise limits as $N \rightarrow \infty$. Equation (10) then follows from strong resolvent convergence of $H_N(t)$ to $H_\infty(t)$ and $\sigma_{\text{ess}}(H_\infty(t)) \supseteq [0, \infty)$ is a consequence of $V_\infty(t, x) \rightarrow_{x \rightarrow +\infty} 0$. Next one constructs the Weyl m -functions $m_\infty^\pm(t, z)$ associated with $H_{\infty, D}^\pm(t)$, the restriction of $H_\infty(t)$ to the interval $(0, \pm\infty)$ with a Dirichlet boundary condition at 0. One obtains

$$\begin{aligned} m_\infty^\pm(t, z) &= \pm i\sqrt{z} \mp [1 \mp i \sum_{j=1}^{\infty} (\sqrt{z} \pm i\kappa_j)^{-1} c_j \psi_{\infty, j}(t, 0)]^{-1} \\ &\quad \times i \sum_{j=1}^{\infty} (\sqrt{z} \pm i\kappa_j)^{-1} c_j [\partial_x \psi_{\infty, j}(t, 0) - \kappa_j \psi_{\infty, j}(t, 0)], \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned}$$

defining the branch of \sqrt{z} by $\lim_{\varepsilon \downarrow 0} \sqrt{|\lambda| \pm i\varepsilon} = \pm|\lambda|^{1/2}$, $\lim_{\varepsilon \downarrow 0} \sqrt{-|\lambda| \pm i\varepsilon} = i|\lambda|^{1/2}$. Since $m_\infty^\pm(t, \cdot)$ are bounded on any region of the type $J_{\delta, R_1, R_2} = \{z = \lambda + i\nu \mid R_1 < \lambda < R_2, 0 < \nu < \delta\}$, $\delta, R_1, R_2 > 0$ and

$$\lim_{\varepsilon \downarrow 0} |\text{Im}[m_\infty^\pm(t, \lambda + i\varepsilon)]|$$

is bounded away from zero for $\lambda \in (R_1, R_2)$, the spectrum of $H_\infty(t)$ in $(0, \infty)$ is purely absolutely continuous by Theorem 3.1 of [11] and hence (9) and $\sigma_{\text{ac}}(H_\infty(t)) \supset (0, \infty)$ follow. Next one proves the following lemma on the basis of H^p -theory, $0 < p < 1$ (see, e.g., [3]).

Lemma 3.3. *Let $\{a_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$, $\{a_j\}_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then there exists a real-valued function f on $[0, \infty)$ with*

$$m(\{x \in [0, \infty) \mid f(x) = c\}) = 0 \quad \text{for each } c \in \mathbb{R}$$

such that

$$\lim_{\varepsilon \downarrow 0} \sum_{j=1}^{\infty} \frac{a_j}{\sqrt{x - i\varepsilon - b_j}} = f(x) \quad \text{for } m\text{-a.e. } x \geq 0.$$

(Here m denotes the Lebesgue measure on \mathbb{R} .) Identifying $x = -\lambda$, $\lambda < 0$, $b_j = \mp \kappa_j$, $a_j = c_j \psi_{\infty, j}(t, 0)$ resp. $a_j = c_j [\partial_x \psi_{\infty, j}(t, 0) - \kappa_j \psi_{\infty, j}(0, t)]$, Lemma 3 applied to $m_{\infty}^{\pm}(t, \cdot)$ yields the existence of real-valued and finite limits of $m_{\infty}^{\pm}(t, \lambda + i\varepsilon)$ for a.e. $\lambda < 0$ as $\varepsilon \downarrow 0$. Thus

$$\mu_{\infty, \text{ac}}^{\pm}((-\infty, 0)) = 0,$$

where $\mu_{\infty, \text{ac}}^{\pm}$ is the absolutely continuous part (with respect to m) of the Stieltjes measure generated by the spectral function of $H_{\infty, D}^{\pm}(t)$. Consequently $\sigma_{\text{ac}}(H_{\infty, D}^{\pm}(t))$, being the topological support of $\mu_{\infty, \text{ac}}^{\pm}$, is contained in $[0, \infty)$. This together with $\sigma_{\text{ac}}(H_{\infty}(t)) = \sigma_{\text{ac}}(H_{\infty, D}^{+}(t)) \cup \sigma_{\text{ac}}(H_{\infty, D}^{-}(t))$ yields

$$\sigma_{\text{ac}}(H_{\infty}(t)) \cap (-\infty, 0) = \emptyset$$

and hence (8). The rest of Theorem 1 is plain.

Finally we turn to Theorem 2. Due to (17), the fact that $\partial_x^2 \psi_{\infty, j} = (V_{\infty} + \kappa_j^2) \psi_{\infty, j}$ and the KdV-equation (6) for V_{∞} one can show it suffices to prove (13) for $0 \leq m \leq 2$ and $n = 0$. This is accomplished in a series of steps. First one proves the crucial identity

$$\int_{\mathbb{R}} dx V_{\infty}(t, x) = -4 \sum_{j=1}^{\infty} \kappa_j,$$

which follows from

Lemma 4. *Assume the hypotheses in Theorem 1. Then*

$$\det_1[1 + C_{\infty}(0, x)] = 1 + \sum_{I \in \mathcal{P}} a_{\infty, I} e^{-2 \sum_{j \in I} \kappa_j x},$$

where \mathcal{P} is the family of all finite, nonempty subsets of \mathbb{N} and $a_{\infty, I} > 0$ are positive numbers (whose precise value turns out to be immaterial for the proof of Theorem 2)

and a detailed study of the asymptotic behavior of $\det_1[1 + C_{\infty}(0, x)]$ as $|x| \rightarrow \infty$.

In the sequel one repeatedly invokes the identity (17) and Vitali's theorem ([9, p. 203]). The rest of Theorem 2 follows from Theorem 1(ii) and scattering theory for $L^1(\mathbb{R})$ -potentials.

Detailed proofs can be found in [7].

We feel that the simplicity of constructing KdV-solutions producing these remarkable spectral (resp. scattering) properties represents a significant result that deserves further investigations. In particular, generalizations, replacing the N -soliton KdV-solutions V_N by N -gap quasi-periodic KdV-solutions and studying the

limit $N \rightarrow \infty$ involving accumulations of spectral gaps and bands, appear to offer a variety of interesting and challenging problems.

Due to the close resemblance of the determinant structure of the N -soliton solutions of (hierarchies of) integrable systems such as the AKNS-class (particularly the nonlinear Schrödinger and Sine-Gordon equations), the Toda lattice, and especially the Kadomtsev-Petviashvili equation, the methods in this paper are by no means confined to KdV-type equations but are widely applicable in this field.

ACKNOWLEDGMENTS

We are indebted to N. Kalton and W. Kirsch for several stimulating discussions. In particular, we would like to thank N. Kalton for invaluable help in connection with Lemma 3.

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