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BOOK REVIEW

Tubes, by Alfred Gray. Addison-Wesley, 1990, 280 pp., \$44.25. ISBN 0-201-15676-8

This book is an outgrowth of a twelve-page article of Hermann Weyl [4], which presents a formula for the volume of the set of points in n-dimensional Euclidean space of distance less than ε from a given m-dimensional compact submanifold M, when ε is small. This set, to be denoted by $M(\varepsilon)$, is called a "tube." Weyl's formula for the volume of tubes played a key role in the first proofs of the generalized Gauss-Bonnet theorem.

The first step in the derivation of Weyl's tube formula makes use of the second fundamental form or, in the terminology of O'Neill, the "shape operator" of M. For a given vector u normal to M at p, the shape operator is the self-adjoint linear map

$$S_u\colon T_pM \to T_pM, \qquad S_u(v) = -(\nabla_v u)^\top,$$

where $\nabla_v u$ is the directional derivative of u in the direction of v and $(\ )^\top$ denotes tangential projection. Weyl showed that if ε is chosen so small that the exponential map is a diffeomorphism from the set of vectors in the normal bundle to M of length less than ε to $M(\varepsilon)$, then

$$V(M(\varepsilon)) = \text{volume of } (M(\varepsilon)) = \int_M \left[\int_{|u|\le\varepsilon} \det(I - S_u)\, du \right] d\mu_M,$$

where $d\mu_M$ is the volume form on M and the inner integral is "integration over the fiber" of $M(\varepsilon) \to M$. (The book under review gives a nice treatment of this formula using the Riccati differential equation for the second fundamental form.)

It is convenient to change variables, obtaining

$$\begin{aligned} V(M(\varepsilon)) &= \int_M \left[\int_{|u|\le 1} \varepsilon^p \det(I - \varepsilon S_u)\, du \right] d\mu_M \\ &= \int_M \left[\int_{|\mu|\le 1} \varepsilon^p [1 - \varepsilon\sigma_1(S_u) + \cdots \pm \varepsilon^m \sigma_m(S_u)]\, du \right] d\mu_M, \end{aligned}$$

where $m = \dim M$, $p = n - m$, and $\sigma_1(S_u), \dots, \sigma_m(S_u)$ are the elementary symmetric functions of the eigenvalues of S_u. Weyl discovered the remarkable fact that, although the integral of σ_k over the fiber vanishes when k is odd, this integral is a polynomial in the curvature of M when k is even. Carrying out the integration leads to Weyl's tube formula

$$V(M(\varepsilon)) = \Omega_p \left[\varepsilon^p + \frac{\varepsilon^{p+2} k_2}{p+2} + \cdots + \frac{\varepsilon^{p+2[m/2]} k_{2_{[m/2]}}}{(p+2)\cdots(p+2[m/2])} \right],$$

where Ω_p is the volume of the unit p-ball and the k_i's are "total curvatures," integrals over M of polynomials in the curvature. For example, k_2 is one-half the integral of the scalar curvature.

In the special case where M has dimension one, this formula states that the volume of $M(\varepsilon)$ is $\Omega_{n-1}\varepsilon^{n-1} \times$ (length of M), a formula that was discovered earlier by Hotelling. In the case of a surface, the Gauss-Bonnet theorem states that $k_2/2\pi$ is the Euler characteristic of M, so the tube volume depends only on the radius ε, the dimension of the ambient space, and the topology of M.

More generally, Allendoerfer and Fenchel discovered that if M is even dimensional then the highest order term in Weyl's tube formula is a topological invariant. Their idea was to use the "Gauss" map $g\colon \partial M(\varepsilon) \to S^{n-1}$, defined by assigning the unit vector u to a point on $\partial M(\varepsilon)$ obtained by moving a distance ε away from M in the normal direction u. The preimage of a given $u \in S^{n-1}$ under g is the set of critical points for the height function

$$h_u \colon \partial M(\varepsilon) \to R \quad \text{defined by } h_u(p) = u \cdot p.$$

For almost all choices of u, h_u is a Morse function. Moreover, its critical points will have even or odd Morse index, according to whether the Gauss map preserves or reverses orientation. Hence the degree of g is

$$\#\text{ of critical points of even index} - \#\text{ of critical points of odd index},$$

which, according to Morse theory, is just the Euler characteristic $\chi(M)$.

On the other hand, the degree of g is just the integral over $\partial M(\varepsilon)$ of $g^*(d\mu_{S^{n-1}})$, where $d\mu_{S^{n-1}}$ is the volume form on the unit sphere S^{n-1}. A direct calculation showed that

$$\int_{\partial M(\varepsilon)} g^*(d\mu_{S^{n-1}}) = \int_M \left[\int_{|u|=1} \sigma_m(S_u)\, du \right] d\mu_M,$$

a constant multiple of the highest order term in Weyl's tube formula. Integration over the fiber thus led Allendoerfer and Fenchel to the Gauss-Bonnet formula for even-dimensional manifolds isometrically embedded in Euclidean space,

$$\chi(M) = \text{degree of } g = \cdots = (2\pi)^{m/2} k_m\,.$$

It was discovered that this argument could be extended to manifolds with boundary—in fact, Allendoerfer and Weil [1] derived a Gauss-Bonnet formula for Riemannian polyhedra, which included as a special case, a formula for the volume of an even-dimensional spherical simplex, due to Poincaré [3].

The one blemish in the argument sketched above, the dependence upon an isometric imbedding in an ambient Euclidean space, was removed by Chern's intrinsic proof of the Gauss-Bonnet formula [2]. Although Chern's proof of the Gauss-Bonnet formula is simpler, the earlier Allendoerfer-Fenchel argument is very concrete and suggests further integral theorems for submanifolds.

The new book by Alfred Gray will do much to make Weyl's tube formula more accessible to modern readers. The first five chapters give a careful and thorough discussion of each step in the derivation and its application to the Gauss-Bonnet formula. Gray's pace is quite leisurely, and a graduate student who has completed a basic differential geometry course will have little difficulty following the presentation.

In the remaining five chapters of the book, one can find an extension of Weyl's tube formula to complex submanifolds of complex projective space, power series expansions for tube volumes, and the "half-tube formula" for hypersurfaces. A high point is the presentation of estimates for the volumes of tubes in ambient Riemannian manifolds whose curvature is bounded above or below. Altogether, I found this book enjoyable to read and can recommend it very highly.

References

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