

BOOK REVIEW

The topology of torus actions on symplectic manifolds, by Michèle Audin. Birkhäuser-Verlag, Basel, Boston, and Berlin, 1991, 181 pp., \$64.00. ISBN3-7643-2602-6, ISBN 0-8176-2602-6

Nowadays it seems to be widely admitted that a mathematical theory reaching its final stage often becomes almost a triviality. The main technical emphasis in such a theory is put on the language, i.e., the notions and definitions, which must be very precise, though often quite complex to cover various applications, and so developed that the proofs turn out simple and clear. A basic example here is the theory of differential forms with its highly elaborated language and very few simple, but fundamental results like Stoke's theorem and Poincaré's lemma. The theory of Hamiltonian actions did originally invent certain very useful notions, e.g., the momentum mapping, but also borrowed some technical tools (Morse theory, equivariant cohomology) from other fields. Now, after a decade of dramatic progress, many chapters of the theory have reached the final stage, and a perfect indication of this is the book by Michèle Audin.

1. CONVEXITY THEOREM

Perhaps the most famous and beautiful result that the theory of Hamiltonian torus actions is known for is the Atiyah-Guillemin-Sternberg convexity theorem [At] and [GS1]:

Let (W, ω) be a compact connected symplectic manifold equipped with a Hamiltonian action of a torus T . Then the image P of the momentum mapping $J: W \rightarrow \mathfrak{t}^$ is a convex polyhedron and all the preimages $J^{-1}(y)$, $y \in \mathfrak{t}^*$ are connected.*

The convexity theorem is a very advanced generalization of the classical result by Schur [S]: the main diagonals of $n \times n$ Hermitian matrices with fixed eigenvalues form a convex set in \mathbb{R}^n . The reader interested in the relation between these two results should consult the book or original papers [At, GS1].

Here I cannot resist the temptation to outline the proof, but first let us recall the definitions and notions we have used. A symplectic manifold is a manifold equipped with a closed 2-form ω which satisfies the nondegeneracy hypothesis: $\omega^n \neq 0$ anywhere on W . Here, $2n = \dim W$. Examples: the cotangent bundle to a manifold and coadjoint orbits of a Lie group carry natural symplectic structures. Any function h on W gives rise to the vector field ξ defined by the Hamilton equation $i_\xi \omega = dh$. The Hamiltonian h is said to be periodic if ξ generates a circle action on W ; a circle action obtained in this way is called Hamiltonian. It is easy

to see that the critical points of h are precisely the fixed points of the action. The next bit of symplectic geometry we need is the Darboux theorem: a symplectic structure can be linearized near every point, i.e., it is locally diffeomorphic to the “standard” linear symplectic structure $\sum dp_i \wedge dq_i$ on \mathbb{R}^{2n} . Moreover, it can be linearized along with a Hamiltonian action of the circle T^1 (the equivariant Darboux theorem): if $x \in W$ is a fixed point of the action then a neighborhood of x is equivariantly diffeomorphic to a neighborhood of the origin in \mathbb{R}^{2n} equipped with the standard symplectic structure and the T^1 -action given by the quadratic Hamiltonian $\sum a_i(p_i^2 + q_i^2)$. This implies that h is a Morse-Bott function (i.e., its critical set is a smooth submanifold and d^2h is nondegenerate on the transversals to the critical set), the critical submanifolds of h are even-dimensional (in fact, symplectic), and all the indexes are even. Now Morse theory tells us that all the levels of h are connected (the convexity theorem for T^1 -actions!) and, in particular, h has a unique local maximum and minimum. If the critical points of h are isolated then h is a perfect Morse function.

Consider now a Hamiltonian action of the torus T on W with the momentum mapping J . Such an action can also be linearized near a fixed point. This argument implies that:

(i) The image $J(P)$ is “locally convex”: for any $x \in W$ there exists a neighborhood U of x and V of $y = J(x)$ such that $J(U) = C_x \cap V$, where C_x is an affine convex polyhedral cone in \mathfrak{t}^* with the vertex at y . (To prove this it is enough to linearize the action of Stab_x .)

(ii) Everything we said about a periodic Hamiltonian remains correct for a quasi-periodic one, i.e., for $h \in C^\infty(W)$ such that the flow of ξ topologically generates a torus action on W . In particular, for any linear function π on \mathfrak{t}^* the function $\pi \circ J$ has a unique local minimum.

(iii) For every $x \in W$ we have $P \subset C_x$. In fact, if P is not contained in C_x , then one could find a linear function π such that $\pi \circ J$ has at least two local minima. But this contradicts (ii)!

Finally, by (i) and (iii) the image P must be a convex polyhedron and, moreover, P coincides with the intersection of all the cones C_x . Arranging this argument in a more elaborate fashion one can prove that the “fiber” of J is connected.

2. EQUIVARIANT COHOMOLOGY AND THE DUISTERMAAT-HECKMAN FORMULA

The stationary phase principle takes an especially simple and explicit form for a periodic Hamiltonian h . Namely, it was proved by Duistermaat and Heckman [DH] that if all the critical points x of h are isolated then we have the exact formula

$$(1) \quad \int_W e^{-\sqrt{-1}ht} \frac{\omega^n}{n!} = \frac{1}{(\sqrt{-1}t)^n} \sum_x \frac{e^{-\sqrt{-1}h(x)t}}{e_x},$$

where the integer number e_x is assigned to a critical point x as follows. Every x is a fixed point of the T^1 -action given by h ; thus, as before we can linearize the action near x and so write $h = h(x) + \sum a_i(p_i^2 + q_i^2)$ in some local coordinates p_i, q_i near x . Then we set $e_x = a_1 a_2 \cdots a_n$. Note that e_x is an integer because h is periodic and so all coefficients a_i must be integers.

A very simple and enlightening proof of this result is due to Atiyah and Bott [AB] and Berline and Vergne [BV]. The key point of the proof is to treat the left-hand

side as the integral of certain equivariant cohomology class. Let $T = T^1$. Following [AB] we identify the T -equivariant cohomology $H_T^*(W)$ with the cohomology of the equivariant de Rham complex. This complex is just the graded ring $\Omega_T^*(W)[u]$ equipped with the differential $D = d + i_\xi u$, where u is an “independent variable,” $\deg u = 2$, and $\Omega_T^*(W)$ is the ring of T -invariant differential forms. For example, if the action is trivial: $\xi = 0$, then $H_T^*(W) = H^*(W) \otimes \mathcal{R}$, where $\mathcal{R} = \mathbb{R}[u]$. Note also that ω is not equivariantly closed, but its “extension” $\omega - h \cdot u$ is.

Pick an equivariant form $\sigma \in \Omega_T^*(W)[u]$ and expand it as $\sum \sigma_i f_i(u)$, where $f_i \in \mathcal{R}$ and σ_i is a differential form of degree i . Now it is natural to set $\int_W \sigma = \int_W \sigma_{2n}$. The localization theorem (see [AB] or the book) says that the integral of σ depends only on the restriction of σ to the fixed point set W^T :

$$(2) \quad \int_W \sigma = \sum_{x \in W^T} \frac{\sigma_0(x) f_0(u)}{e_x u^n}.$$

To prove (1) we apply (2) to $\sigma = \exp(\omega - h \cdot u)$ and set $u = \sqrt{-1}t$.

Among various applications of (2) we would like to mention just one more, namely, the relation

$$(3) \quad \sum_{x \in W^T} \frac{1}{e_x} = 0,$$

which, as well as (2), holds for a symplectic, i.e., ω -preserving, or even almost-complex T -action. To prove it one has to substitute $\sigma = 1$ in (2). For a Hamiltonian action (3) also follows from (1) [DH].

The original proof of (1) is based on another argument, though it is also relevant to equivariant cohomology. Namely, the right-hand side of (1), regarded as a function of t , is the Fourier transform of the function f such that for any regular value c of h we set $f(c)$ equal to the volume of the level $M_c = \{h = c\}$, i.e., $f(c)$ is the integral of ω^{2n}/dh over M_c . According to Duistermaat and Heckman [DH], f is a piecewise polynomial function. To see this we equivariantly identify all the levels M_c , $c \in I = (c_0, c_1)$, where the interval I does not contain critical points of h . Thus the reduced symplectic structure ω_c on $B_c = M_c/T$, becomes a function of c . A straightforward calculation shows that this function is linear (i.e., affine), so $f(c) = \int_{B_c} \omega_c^{n-1}$ must be polynomial on I .

Coming back to equivariant cohomology we observe that $H_T^*(M_c) = H^*(B_c, \mathbb{R})$ because the T -action on M_c is locally free and the equivariant cohomology is taken over \mathbb{R} . It turns out that the derivative $\chi = d[\omega_c]/dc$ has a nice topological interpretation. Namely, the cohomology class χ is the so-called Euler class of the locally free action on M_c (independent of $c \in I$). For example, if the action is free, then χ is just the Euler class of the circle bundle $M_c \rightarrow B_c$. Motivated by this simple analysis, we come to the question of what happens to B_c , χ , and ω_c when c goes through a critical value of h . The book under review contains a detailed account on this matter (see also [M, G] and, in particular, [GS2]).

As we see equivariant cohomology is a useful and powerful tool to be used in the theory of Hamiltonian actions. But what makes it especially attractive is that $H_T^*(W)$ is easy to calculate. Namely, for a Hamiltonian action $H_T^*(W)$ is isomorphic to $H^*(W) \otimes \mathcal{R}$ as an \mathcal{R} -module. In other words the cohomology does not feel the

action at all, i.e., it behaves as if the action were trivial. This result is apparently due to Kirwan [K1].

The proof is especially simple for a circle action with isolated fixed points. Analyzing the Morse complex of h it is not hard to show that any closed T -invariant k -form σ_k admits a D -closed “extension” $\sigma = \sigma_k + \sigma_{k-2}u + \sigma_{k-4}u^2 + \cdots$, where $\sigma_i \in \Omega_T^i(W)$. Moreover, the extension σ can be made canonical if, for example, we employ the Hodge theory. Thus we have an \mathbb{R} -linear monomorphism $H^*(W) \rightarrow H_T^*(W)$, which, being \mathcal{R} -linear extended, gives rise to the desired isomorphism $H^*(W) \times \mathcal{R} \rightarrow H_T^*(W)$.

Unfortunately this simple method does not work that well when the fixed points are not isolated. However, applying a more advanced technique, one can prove the general case literally in a few lines. Recall that there exists a spectral sequence with $E_2^* = H^*(W) \otimes \mathcal{R}$ converging to the equivariant cohomology and thus $\text{rk}_{\mathcal{R}} H_T^*(W) \leq \dim H^*(W)$. On the other hand, by the Borel localization theorem [B] (see also the book or [AB] for a modern treatment) $\text{rk}_{\mathcal{R}} H_T^*(W) = \dim H^*(W^T)$. Finally, since W^T is the critical set of a quasi-periodic Hamiltonian, $\dim H^*(W^T) \geq \dim H^*(W)$ and, as a consequence, $\text{rk}_{\mathcal{R}} E_2^* = \text{rk}_{\mathcal{R}} H_T^*(W)$. This only can happen when the spectral sequence degenerates and the E_2 -term coincides with the equivariant cohomology. A rather formal argument shows that the same is true for a Hamiltonian action of any compact group. Moreover, this result (properly reformulated), as well as the Duistermaat-Heckman formula, remains correct for some symplectic actions [G].

3. MORE ABOUT THE BOOK

The book by Michèle Audin appears to be a perfect self-contained introduction into the theory of Hamiltonian torus actions. The first two chapters contain some preliminaries on compact group actions on manifolds (e.g., the slice theorem) and symplectic geometry. The topics we have discussed are covered in detail in Chapters 3 and 5, respectively, together with all the necessary technique (Morse theory and equivariant cohomology). In Chapter 5 the reader will be pleased to find a simple and intuitive definition of equivariant cohomology via the equivariant de Rham complex, which complements the standard approach based on formal, though useful and very powerful, algebraic topology constructions.

Chapter 4 is entirely devoted to the Hamiltonian actions on compact 4-manifolds, in particular, circle actions. Such actions occupy a very special place in the theory, because, on the one hand, the first nontrivial examples are among them, and on the other hand, these actions can be effectively described [Au]. Namely, every such action can be obtained from one of a small number of basic examples by a sequence of blow-ups. Another interesting result we have in this dimension is that according to a theorem of McDuff [M] a symplectic circle action on a compact connected 4-manifold is Hamiltonian if and only if it has fixed points. Proof [G]: Consider a non-Hamiltonian action; for the sake of simplicity we assume that the form $\alpha = i_{\xi}\omega$ is integral. Then the Morse theory argument shows that all the critical points x of α are isolated and have index 2, thus $e_x < 0$. By (3) this is possible only if $W^T = \emptyset$. Conversely, it is clear that a Hamiltonian action does have fixed points, namely, critical points of a Hamiltonian.

The last chapter concerns toric varieties, which appear in this context as symplectic manifolds with the biggest Abelian compact group of symmetries. In other

words a compact symplectic $2n$ -manifold is toric if it admits a Hamiltonian T^n -action. (Note that here toric manifolds carry symplectic rather than a complex structure.) Due to Delzant [D] such a manifold is uniquely determined by the image P of the momentum mapping. Thus the topology of the manifold and the action can be recovered from the combinatorics of the polyhedron P . Reading this relatively elementary chapter requires considerable effort, in part, because some of the results are unclear or even incorrectly stated. (In particular, this concerns §3.) Hence I hardly can recommend it as an introduction in toric varieties. On the other hand, the approach developed here, in certain aspects, is much easier than the standard treatment of toric varieties. Thus I believe that experts will find the material presented in Chapter 6 interesting and very useful.

The book under review is by no means a monograph but rather a textbook. In other words *Topology of torus actions on symplectic manifolds* does not intend to cover the entire field, and some, even important, results are just briefly mentioned. Among such results are, for instance, the calculation of equivariant cohomology (outlined in the second section of this review) and the deep convexity theorem for non-Abelian groups due to Kirwan [K2]. (As the title indicates, the book is on torus actions.)

The book is written in an elegant and informal style, but not at the cost of details, and with a care about technicalities. Numerous, often very instructive, exercises and examples, as well as excursions into related areas, give the reader a chance to learn not only the theory and applications of Hamiltonian torus actions, but what may even be more important, its place in the complex conglomerate of modern mathematics.

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