

BOOK REVIEW

The energy method, stability, and nonlinear convection, by Brian Straughan, Springer-Verlag, New York, 1992, xii+242 pp., \$49.80. ISBN 0-387-97658-2

Suppose that $f(\cdot)$ is a nonnegative, differentiable function defined on $I = [t_0, \infty)$ and that $f'(t) \leq a(t)f(t)$ for some locally integrable function $a(\cdot)$ on I . Then

$$0 \leq f(t) \leq f(\tau) \exp\left(\int_{\tau}^t a(\eta) d\eta\right)$$

for $t_0 \leq \tau \leq t < +\infty$. This observation is called Gronwall's Inequality. If $\int_{t_0}^{+\infty} a(\eta) d\eta = -\infty$, then $\lim_{t \rightarrow +\infty} f(t) = 0$.

The book under review is concerned with applications of this simple principle to questions of stability of solutions of problems in fluid dynamics. These problems are *not* simple.

The application of this principle to the fluid dynamical problems discussed in Straughan's book is often called the Energy Method and is closely related to the Lyapunov method. These problems have a rich mathematical and physical structure and are among the most important problems in applied mathematics. We shall mention a few of them later in this review.

The underlying feature of all of these problems is that the fluid motion itself is the primary mechanism which drives the transport of various physical quantities such as momentum, temperature, salinity, etc. This transport by the fluid motion is called convection. In this context, a familiar equation, the heat equation for the temperature T with thermal diffusivity D ,

$$\frac{\partial T}{\partial t} = D \sum_{i=1}^3 \frac{\partial^2 T}{\partial x_i^2},$$

which describes the transport of temperature in a solid body without internal sources of heat, must be modified to include the effect of fluid motion. It becomes

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial T}{\partial x_i} = D\Delta T$$

where $\vec{u} = (u_1, u_2, u_3)(\vec{x}, t)$ is the velocity of the fluid of a fixed point \vec{x} , in space and time t . The operator D/Dt is called the material derivative. The extra term, $\vec{u} \cdot \vec{\nabla} T$, is called the convective term or, in the case of momentum transport, the

convective acceleration. This must be added to include the effect of velocity transport of a physical quantity such as heat in a fluid such as water, oil, or air. (The classical Navier-Stokes equations describe the transport of momentum in a viscous, incompressible fluid, such as water or motor oil.)

In order to illustrate the principle focus of this book in a relatively jargon-free context without recourse to specific examples, we consider the abstract (usually nonlinear) differential equation

$$(1) \quad du/dt = A(u(t))$$

where $u: I \rightarrow D(A) \subset H$, where H is a Hilbert space. The set $\{U \in D(A) | A(U) = 0\}$ is called the set of stationary solutions or, in the language of fluid dynamics, the set of basic states. Straughan's book is concerned with the asymptotic stability of such basic states in some norm ($\|\cdot\|$) related to the norm on H and the operator A .

Consider, for example, a fixed stationary solution U and write $u(t) = v(t) + U$. If $\|v(t)\| \rightarrow 0$ as $t \rightarrow +\infty$ for *all* choices of $v(t_0)$, U is said to be globally asymptotically stable. If there is $\delta > 0$ such that $\|v(t_0)\| < \delta$ implies $\|v(t)\| \rightarrow 0$, then U is locally asymptotically stable. If for every $\delta > 0$, there is $v(t_0)$ with $\|v(t_0)\| < \delta$, such that if $v(\cdot)$ exists on (t_0, ∞) then $\liminf_{t \rightarrow +\infty} \|v(t)\| > 0$ or else $v(\cdot)$ does not exist on $[t_0, \infty)$, we say U is unstable.

The idea of the energy method to prove stability is really quite easy. One defines some functional $E(v)$ of the perturbation v and sets $f(t) = E(v(t))$ and tries to show that f satisfies Gronwall's inequality with $\int^{+\infty} a(t) dt = -\infty$. $E(v)$, the energy, will have the properties of a norm, namely, $E(v) > 0$ for $v \neq 0$, $E(0) = 0$ and will be (to within a constant) subadditive. A stability result of this kind can be established for the motion of the coffee in a cup which is left unconsumed after having been poured. The energy will be $\int |\vec{u}|^2 dx$. Here the obvious basic state is $\vec{U} = \vec{0}$. If $A(\cdot)$ has an operator derivative in a suitable sense, then we may also consider the linearized equation about $u = U$:

$$(2) \quad dV/dt = A'(U)V(t)$$

where $A'(U)$ is the operator (Fréchet) derivative evaluated at $u = U$. If solutions of (2) decay to zero for *all* choices of initial values $V(t_0)$ we say that (1) is *linearly stable*. This will be the case if, for example, for some $\lambda > 0$,

$$(x, A'(U)x) \leq -\lambda(x, x) \quad (x \in D(A'(U)))$$

where (\cdot, \cdot) is the scalar product on H . (Then $f(t) \equiv (V(t), V(t))$.)

It is not hard to see that local asymptotically stable solutions are linearly stable. The converse is, however, false in general.

In many applications, A depends upon a parameter R (or even a set of parameters, $\{R_1, R_2, \dots, R_k\}$). For example, in the case of the Navier-Stokes equations, R will be the Reynolds number, a dimensionless ratio of momentum to viscosity. The larger the Reynolds number, the less viscous is the fluid. Compare the ease of flow of molasses and water. Molasses, being more viscous than water, flows less readily and thus has a lower Reynolds number for a given flow rate. There is typically a critical value of R , Re (depending on the basic state) such that if $R < Re$ then U is stable while if $R \geq Re$ then U is unstable. When A depends on R , so does A' , and

there is a critical value R_L ($\geq R_E$) also depending on U with the same property; i.e., if $R < R_L$ (1) is linearly stable at U (if $R > R_L$ it is linearly unstable at U).

When $R_E < R_L$, for any $R \in (R_E, R_L)$ one speaks of the existence of so-called subcritical instabilities.

Under some circumstances, as in the case of steady nonconstant flow of a fluid, this inequality is strict. This is the case, for example, in the flow of fluid between two rotating cylinders. See [1, Chapter 5]. The basic state here is called Couette flow; however, in other cases, $R_E = R_L$. One such situation where this occurs is in the classical Bénard problem. In this problem, a fluid in a flat pan such as spermaceti oil as in Bénard's original experiment is gently heated from below. At first, the heat is transported to the surface solely by conduction; however, as the fluid near the heating elements warms, it will become less dense than the cooler fluid above it. Under the influence of gravity, the heavier, colder fluid falls while the more buoyant lighter, warmer fluid rises. Distinct patterns of cellular fluid motion are observed. (See [1] for some enlightening photographs.) The critical parameter here is R_a , the Rayleigh number. For this problem $R_L = R_E = R_a$. This is due (primarily) to the fact that for the basic state for this problem, the linear operator is symmetric.

(The onset of instability in the Bénard problem is not caused solely by buoyant forces as I have intimated above. In fact, when the depth of the fluid is fairly small, surface tension effects must also be taken into account in determining the onset of instability. For sufficiently thin layers of fluid, it is the variation in surface tension that is the cause of the instability and not buoyancy effects. See [3] for a more complete discussion of this. The text under review also discusses this in Chapter 8.)

In fluid problems, this symmetry, or lack of it, is caused by the absence or presence of convective terms in the differential equations for the basic state. In steady, nonconstant flow these terms are present in the linearization $A'(U)$, while in the Bénard problem, the velocity components of the basic flow vanish and there is only a constant, nonzero temperature gradient in the vertical direction.

In this book Straughan discusses several interesting fluid flow problems in the above framework. These include problems involving penetrative convection, surface tension driven convection, convection in micropolar fluids, and chemically driven convection. In all of these problems, the ideas are as above. Identify the differential equations for the basic states and crucial physical parameter(s) whose critical value(s) determine the onset of instability and subcritical instability. Then calculate these critical numbers numerically or obtain good bounds for them.

In addition to being a very readable introduction to the subject, the book contains considerable current information and reports on several recent results of the author and his coworkers. Although the subject is notorious for its jargon and use of mind-boggling notation, Straughan has managed to keep both the jargon and the notation to tolerable limits. The development of the subject is logical and well planned with one major exception: It would have been more logical to give the abstract definitions of stability and linear stability found on page 62 near the beginning of the book and to apply these definitions to the first simple but instructive examples in Chapter 2. A minor comment: It would have been nice to see some photographs of actual experiments as in [2–4] to illustrate some of the more recent phenomena discussed in later chapters. The illustrations have a way of grabbing the reader's attention and stimulating his/her curiosity in a way that no equation

can!

The book is otherwise well conceived and well written. Even the price, by today's standards, is reasonable. There are roughly 250 literature citations, which give the reader a fairly complete survey of the recent literature on the subject. This is a book that should be on the bookshelf of anyone with a developing interest in fluid mechanics.

REFERENCES

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