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KELLER'S CUBE-TILING CONJECTURE IS FALSE IN HIGH DIMENSIONS

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Abstract. O. H. Keller conjectured in 1930 that in any tiling of $\mathbb{R}^n$ by unit n-cubes there exist two of them having a complete facet in common. O. Perron proved this conjecture for $n \leq 6$. We show that for all $n \geq 10$ there exists a tiling of $\mathbb{R}^n$ by unit n-cubes such that no two n-cubes have a complete facet in common.

1. Introduction

In 1907 Minkowski [5] conjectured that all the extremal lattices for the supremum norm were of a certain simple form and observed that this conjecture had a geometric interpretation: in any lattice tiling of $\mathbb{R}^n$ with unit n-cubes there must exist two cubes having a complete facet ($(n-1)$-face) in common. He proved this for $n = 2$ and 3. In studying this question, Keller [4] generalized it to conjecture that any tiling of $\mathbb{R}^n$ by unit n-cubes contains two cubes having a complete facet in common. In 1940 Perron [6] proved Keller's conjecture for dimensions $n \leq 6$. Soon after, Hajós [2] proved that Minkowski's original conjecture is true in all dimensions. Keller's stronger conjecture remained open. Hajós [3] later gave a combinatorial problem concerning factorization of abelian groups, which he proved was equivalent to Keller's conjecture. Stein [7] gave a survey of these results and other related tiling problems.

More recently, Szabó [8] showed that if Keller's conjecture is false in $\mathbb{R}^n$, then there exists a counterexample tiling in some $\mathbb{R}^m$ (with $m \geq n$) having the following extra properties: the centers of all cubes are in $\frac{1}{2}\mathbb{Z}^m$, and the tiling is periodic with period lattice containing $2\mathbb{Z}^m$. Corrádi and Szabó [1] studied a graph-theoretic version of this latter problem, showing directly that there are no such counterexamples for $m \leq 5$.

We explicitly construct a counterexample tiling of Szabó's type in $\mathbb{R}^{10}$. Keller's conjecture is then false for all $n \geq 10$ because a counterexample tiling in $\mathbb{R}^n$ gives one in $\mathbb{R}^{n+1}$ by "stacking" layers of this tiling with suitable translations made between adjacent layers.

2. Main result

We prove the following result.

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Theorem A. *For $n = 10$ and 12 there exists a tiling of $\mathbb{R}^n$ by unit cubes such that*

(1) *The centers of all cubes are in $\frac{1}{2}\mathbb{Z}^n$;*
(2) *The tiling is periodic with period lattice $2\mathbb{Z}^n$;*
(3) *No two cubes have a complete facet in common.*

Before giving the constructions, we describe Corrádi and Szabó's equivalent graph-theoretic criterion for such a tiling to exist in $\mathbb{R}^n$.

Scale everything up by a factor of 2 to consider tilings of $\mathbb{R}^n$ by translates of the cube

$$\mathsf{C} = \{(x_1, \ldots, x_n) : -1 \le x_i \le 1 \text{ for all } i\}$$

of side 2 centered at the origin, such that the centers of all cubes are in $\mathbb{Z}^n$, and the tiling is periodic with period lattice $4\mathbb{Z}^n$. There are 2^n equivalence classes of cubes $\mathbf{m} + \mathsf{C} + 4\mathbb{Z}^n$ in such a tiling, and each equivalence class contains a unique cube with center

$$\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n, \qquad 0 \le m_i \le 3\,. \tag{1}$$

The collection $\mathcal{S}$ of these 2^n vectors describes the tiling.

Now form two graphs G_n and G_n^*, each of which has 4^n vertices labeled by the 4^n vectors in $\mathbb{Z}^n$ of form (1), as follows. Consider the conditions:

(a) $\mathbf{m}$ and $\mathbf{m}'$ have some $|m_i - m_i'| = 2$.
(b) $\mathbf{m}$ and $\mathbf{m}'$ differ in two coordinate directions.

G_n has an edge between vertices $\mathbf{m}$ and $\mathbf{m}'$ if (a) holds, while G_n^* has an edge between $\mathbf{m}$ and $\mathbf{m}'$ if (a) and (b) both hold. Condition (a) says that all translates under $4\mathbb{Z}^n$ of cubes C centered at $\mathbf{m}$ and $\mathbf{m}'$ have disjoint interiors, while (a) and (b) together say that all translates under $4\mathbb{Z}^n$ of such cubes also do not have a complete facet in common.

A set $\mathcal{S}$ of 2^n vectors satisfying (1) yields a $4\mathbb{Z}^n$-periodic cube tiling if and only if $\mathcal{S}$ forms a clique in G_n and it yields a $4\mathbb{Z}^n$-periodic cube tiling with no two cubes having a complete facet in common if and only if $\mathcal{S}$ forms a clique in G_n^*. This gives the Corrádi-Szabó criterion that a Szabó-type counterexample exists in $\mathbb{R}^n$ if and only if G_n^* contains a clique of size 2^n.

The graph G_n is the complement of the product graph $C_4 \otimes C_4 \otimes \cdots \otimes C_4$ of n copies of the 4-cycle C_4. It has maximal clique size equal to the independent set number of $C_4 \otimes \cdots \otimes C_4$, which is $\alpha(C_4)^n = 2^n$ since C_4 is a perfect graph and $\alpha(C_4) = 2$. In fact, G_n has an enormous number of maximal cliques, and the problem is whether or not any of them remain a clique in G_n^*.

Note also that the graphs G_n and G_n^* have large groups of automorphisms. On both graphs one can relabel the vertices $\mathbf{m} = (m_1, \ldots, m_n)$ by relabeling the ith coordinate using the group generated by the cyclic permutations (0123) and the 2-cycle (13), and one also can permute coordinates. This generates a group of $8^n n!$ automorphisms.

Proof of Theorem A. We give the easier 12-dimensional construction first. It starts with the set $\mathcal{T}$ of vectors given in Table 1, which (ignoring the primes on some zeros) is a clique of size 8 in G_3. It is very nearly a clique for G_3^*, in that it omits

TABLE 1. Clique $\mathcal{T}$ in G_3.

0	0	0
2	0	1
1	2	0
0	1	2
2	0′	3
3	2	0′
0′	3	2
2	2	2

only three edges, namely, 201–20′3, 120–320′, 012–0′32. If somehow $0'$ were distinct from 0, while it was still the case that $2 - 0' = 2$, then this would be a 2^3-clique for G_3^* and would give a counterexample.

We use a block substitution construction that in effect accomplishes this in $\mathbb{R}^{3k}$ for suitable k. Assign to each of $0, 0', 1, 2, 3$ sets S_0, S_0', S_1, S_2, S_3 of vectors in $\{0,1,2,3\}^k$ having the following properties:

(i) Each of S_0, S_0', S_1, S_2, S_3 is a clique in G_k^*.
(ii) No two of the sets S_0, S_0', S_1, S_2, S_3 have a common vector.
(iii) $S_0 \cup S_2, S_0' \cup S_2$, and $S_1 \cup S_3$ are each a clique in G_k.

Assuming (i), (ii), the last condition (iii) says, e.g., for $S_0 \cup S_2$ each element of S_0 differs from each element of S_2 by 2 (mod 4) in some coordinate.

Call the vectors in the sets S_i blocks. Form the set $\mathcal{S}$ of all vectors in $\{0,1,2,3\}^{3k}$ that can be formed by taking any vector $(m_1, m_2, m_3) \in \mathcal{T}$ and for each m_i substituting any block in the corresponding S_{m_i}, independently for each i.

Claim. $\mathcal{S}$ is a clique in G_{3k}^*.

To prove this, let $\mathbf{v}, \mathbf{v}'$ be distinct elements of $\mathcal{S}$ constructed from $\mathbf{m} = (m_1, m_2, m_3)$ and $\mathbf{m}' = (m_1', m_2', m_3')$ in $\mathcal{T}$, respectively. If $\mathbf{m} = \mathbf{m}'$ then $\mathbf{v}, \mathbf{v}'$ have some block $\mathbf{w}, \mathbf{w}' \in S_{m_i}$ where they differ, and condition (i) forces an edge between $\mathbf{v}$ and $\mathbf{v}'$ in G_{3k}^*. If $\mathbf{m} \neq \mathbf{m}'$ then $\mathbf{m}$ and $\mathbf{m}'$ differ by 2 in some coordinate (here $0'$ and 2 are considered to differ by 2), which carries over to $\mathbf{v}$ and $\mathbf{v}'$ by condition (iii), and $\mathbf{m}$ and $\mathbf{m}'$ also differ in another coordinate, where 0 is treated as distinct from $0'$, and this carries over to $\mathbf{v}$ and $\mathbf{v}'$ by condition (ii), proving the claim.

If one can choose $|S_0| = |S_0'| = a$, $|S_1| = b$, $|S_2| = c$, and $|S_3| = d$ with $a + c = 2^k$, $b + d = 2^k$, then $|\mathcal{S}| = a^3 + 3abc + 3acd + c^3 = 2^{3k}$ will be a clique in G_{3k}^*, thus giving a counterexample.

We achieve this with $k = 4$, $a = b = 12$, $c = d = 4$, with the sets S_0, S_0', S_1, S_2, S_3 given in Table 2 below. The sets S_0, S_0', S_2 were obtained from a 28-clique for G_5^* given in [1, Table 2]. Examining the first column of this 28-clique, one finds twelve vectors each having value 0 and 2 and four having value 1. Deleting this column and grouping the resulting $\mathbb{Z}^4$-vectors as S_0, S_0', S_2, the G_5^*-clique property guarantees that all the conditions (i), (ii), (iii) that concern only S_0, S_0', S_2, automatically hold. Next we apply a suitable automorphism of G_4^* to S_0, S_2 to obtain S_1, S_3. For *any* automorphism conditions (i) and (iii) will automatically hold for S_1, S_3 obtained this way. Thus we need only to find an automorphism where (ii) holds. The automorphism that cyclically permutes the labels of the first coordinate $0 \to 1 \to 2 \to 3 \to 0$ gives suitable S_1, S_3, as listed in Table 2.

TABLE 2. Blocks used in constructions.

S_0	S_0'	S_2	S_1	S_1'	S_3
0000	0303	0211	1000	1303	1211
0012	1011	1132	1012	2011	2132
0213	1113	2303	1213	2113	3303
0230	1130	3020	1230	2130	0020
0332	1323		1332	2323	
1020	1331		2020	2331	
2100	2211		3100	3211	
2112	3001		3112	0001	
2220	3022		3220	0022	
2301	3103		3301	0103	
2322	3223		3322	0223	
3132	3231		0132	0231	

The conditions (i), (ii), (iii) can be verified directly for S_0, S_0', S_1, S_2, S_3 by hand calculation. Aside from the distinctness of all elements, the automorphism sending (S_0, S_2) to (S_1, S_3) means that one need only check properties for S_0, S_0', S_2. The calculation can be further reduced by observing that there is an automorphism of G_4^* that fixes S_2 and sends S_0 to S_0'. This automorphism cyclically permutes the labels of the first coordinate $0 \to 1 \to 2 \to 3 \to 0$ and the last coordinate $0 \to 3 \to 2 \to 1 \to 0$, and then exchanges these coordinates. Thus one need only verify that S_0 and S_2 are G_4^*-cliques and $S_0 \cup S_2$ is a G_4-clique.

The 10-dimensional construction is similar in nature and is based on the fact that the set $\widetilde{\mathcal{T}} = S_0 \cup S_2$ from Table 2 is a clique of size 2^4 in G_4, which is very nearly a 2^4-clique for G_4^*. In G_4^* it omits only the four edges 0213–0211, 3132–1132, 2301–2303, 1020–3020. Now regard S_2 as being

$$\begin{array}{cccc} 0 & 2 & 1' & 1 \\ 1 & 1' & 3 & 2 \\ 2 & 3 & 0' & 3 \\ 3 & 0' & 2 & 0 \end{array}$$

where we want $0 \neq 0'$ and $1 \neq 1'$. Assign to $0, 0', 1, 1', 2, 3$ the sets of blocks $S_0, S_0', S_1, S_1', S_2, S_3$ in Table 2, where S_1' is constructed from S_1 similarly to S_0' from S_0. These sets satisfy:

(i) Each of $S_0, S_0', S_1, S_1', S_2, S_3$ is a clique in G_4^*.
(ii) No two of these sets have a common vector.
(iii) $S_0 \cup S_2, S_0' \cup S_2, S_1 \cup S_3$, and $S_1' \cup S_3$ are each a clique in G_4.

Apply the block substitution construction to the second and third columns only on $\widetilde{\mathcal{T}}$ to obtain a set $\widetilde{\mathcal{S}}$ of 2^{10} 10-vectors. This is a clique in G_{10}^*, as required. Note that only the second and third columns need to be expanded in blocks, because the primed elements in S_2 above appear only in these columns. □

3. DISCUSSION

The failure of Keller's Conjecture in high dimensions illustrates the general phenomenon that Euclidean space allows more freedom of movement in high dimensions than in low ones. It is interesting that the critical dimension where Keller's Conjecture first fails, which is at least 7, is as high as it is.

It may be a difficult matter to determine exactly the critical dimension. Exhaustive search for Szabó-type counterexamples already seems infeasible for G_7^*; the maximum clique problem is a well-known NP-complete problem, which is also computationally hard in practice. The authors ruled out the existence of any 2^7-clique in G_7^* that is invariant under a cyclic permutation of coordinates by computer search. It is conceivable that there exist Szabó-type counterexamples in dimension 7, 8, or 9, which are all so structureless that they will be hard to find. In any case we have so far found no variant of the constructions of Theorem A that work in these dimensions.

A natural extension of Keller's conjecture is to determine the largest integer K_n such that every tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that have a common face of at least dimension K_n. For a Szabó-type tiling, two cubes having coordinates $(m_1, \ldots, m_n)$ and $(m_1', \ldots, m_n')$ in G_n^* have a k-dimensional face in common if $|m_i - m_i'| = 0$ or 2 for all i, and exactly k values $|m_i - m_i'| = 0$. The 10-dimensional and 12-dimensional cube tilings $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ constructed in Theorem A each contain two cubes sharing a common face of codimension 2, so they imply only $K_{10} \leq 8$ and $K_{12} \leq 10$. We have found a different 10-dimensional cube tiling (using a similar construction) which shows that $K_{10} \leq 7$. We also can show that $n - K_n \to \infty$ as $n \to \infty$; details will appear elsewhere.

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