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BOOK REVIEW

Introduction to complex analytic geometry, by Stanislaw Łojasiewicz. Translated from Polish by Maciej Klimek. Birkhäuser-Verlag, Basel, 1991, xiv+523 pp., \$118.00. ISBN 3-7643-1935-6, ISBN 0-8176-1935-6

Half a century ago, the words "analytic geometry" were used for the treatment of elementary geometrical problems in which the data were represented by real or complex numbers and straightforward computations substituted for astute arguments; a typical example of such "complex analytic geometry" is the following problem. Given the points A, B, C (represented by the complex numbers a, b, c) in the plane, let A', B', C' be such that the triangles BCA', CAB', ABC' are isosceles, have right angles at A', B', C' respectively, and either none of them overlap the triangle ABC or all of them overlap it; then compare the line segments $AA', B'C'$.

As for the word "Introduction" in the title of the book, it means, just as it did for the famous "Introduction to complex analysis" by Hörmander, that the readers are led very far into the subject by an outstanding guide. The latter introduction [Hö], which appeared in 1966, has ever since been the reference book for anybody working on pseudoconvexity or analytic sheaves; Łojasiewicz's book, published in Polish in 1988, needed only an english translation in order to become the reference book for the properties of *analytic sets*, i.e., sets locally defined as sets of common zeros of finitely many analytic functions.

Although he is well known for his fundamental work on real semianalytic sets [Ł], Łojasiewicz deals only with the complex case, on account of its greater richness, illustrated by such facts as these. If S is an analytic set in an open set U in $\mathbf{C}^m$ and S^0 is the set of regular points of S (i.e., points near which S is a manifold), then $S\backslash S^0$ is an analytic set in U; if moreover S is irreducible, then S^0 is connected and has a constant dimension. All of this fails in the real case, e.g., for

$$S = \{(x, y, z) \in \mathbb{R}^3 \colon x^3 + y^2 z + x^2 = 0\},$$

since then $S\backslash S^0$ is the half-axis $\{(0, 0, z) \colon z \leq 0\}$, while S^0 consists of the 1-dimensional set $\{(0, 0, z) \colon z > 0\}$ and a disjoint 2-dimensional set.

The book begins with three chapters: A. Algebra; B. Topology; C. Complex analysis; they contain every point of usefulness for the sequel, but omit any fact of no use, however important by itself (e.g., Hartogs's theorem on separate analyticity), that can be found elsewhere ([BM] or [Hö]). The main topic of Chapter C is the Weierstrass *preparation theorem* (with its division version), whose consequences, presented in Chapters I, II, are: (a) a simple (i.e., irreducible) 1-dimensional germ of analytic set enjoys a homeomorphic parametric description; (b) the factorial and noetherian characters of rings of polynomials go over to the ring $\mathcal{O}_a$ of germs of analytic functions at the point a.

The *local description* of a simple k-dimensional germ A_a of analytic sets at the point a is much more difficult and brings out an interplay between geometric and algebraic phenomena that, as the author writes in his preface, "constitutes the main attraction of the complex theory." To begin with, the germ A_a is characterized by the ideal $I(A_a)$ in $\mathcal{O}_a$ consisting of the germs at a of functions vanishing on A_a; A_a is simple if and only if $I(A_a)$ is prime; the properties of A_a and $I(A_a)$ are mixed up in the choice of an adequate basis for the ambient space $\mathbf{C}^m$. An open set Ω in the subspace generated by the first k elements of this adequate basis, a $(k-1)$-dimensional analytic set Z in Ω, and p holomorphic maps $\Omega \backslash Z \to \mathbf{C}^{m-k}$ can be found such that the closure in $\Omega \times \mathbf{C}^{m-k}$ of the union of the p graphs is an analytic set with precisely the given germ A_a. Resorting to the primitive element theorem, proved in Chapter A (for the author never relies on so-called well-known facts), makes the construction much simpler than it was in [He].

Among the many problems to which this local description yields a good approach, two beautiful theorems give striking features to the noetherian character of the rings $\mathcal{O}_a$. (1) If S is the set of common zeros of the finitely many func- tions f_j, holomorphic on the open set U, and A_a is the germ of S at the point $a \in S$, then $I(A_a)$ does not, in general, coincide with the ideal J_a in $\mathcal{O}_a$ generated by the germs of the f_j at a; actually $I(A_a) = \operatorname{rad} J_a$ and, therefore, $[I(A_a)]^n \subset J_a$ for large enough n—this is known as Hilbert's Nullstellensatz. (2) By a *coherence theorem* of H. Cartan, finitely many functions g_k, holomorphic on an open neighborhood V of a, can be chosen so that their germs at *each* point $x \in S \cap V$ generate the ideal $I(A_x)$ in $\mathcal{O}_x$.

Somewhat less classical are the contents of Chapters IV, V, VI of the book: the Thullen-Remmert-Stein theorem on the continuation of a k-dimensional analytic set in the complement of an analytic set of dimension $< k$; Remmert's proper mapping theorem and rank theorem, both furnishing sufficient conditions for the image, under a holomorphic map $X \to Y$, of an analytic set in X, to be analytic in Y. Here X and Y may be *analytic spaces* (first named complex spaces when they appeared in the literature), i.e., Hausdorff spaces locally modeled on analytic subsets of finite-dimensional vector spaces, just the same way as complex manifolds are locally modeled on open subsets of these vector spaces.

A point a in the analytic space X is said to be normal if every bounded holomorphic function on $X^0 \cap$ (an open neighborhood of a) extends holomorphically to this neighborhood, or, equivalently, if the ring $\mathcal{O}_a$ is an integrally closed integral domain; an example of a nonnormal point is the origin for the analytic set $\{(x,y) \in \mathbf{C}^2 \colon y^2 = x^3\} = \{(t^2,t^3)\colon t \in \mathbf{C}\}$. Any analytic space X admits a normalization (constructed in Chapter VI), i.e., a proper holomorphic map $\pi\colon Y \to X$ with the properties: Y is an analytic space with normal points only, $\pi^{-1}(X^0)$ is dense in Y, $\pi^{-1}(x)$ is a finite set for all $x \in X$ and a singleton for all $x \in X^0$; in the above example, $Y = \mathbf{C}$, $\pi(t) = (t^2, t^3)$ for all $t \in \mathbf{C}$.

The book ends with a long seventh chapter of particular interest, for it presents, with complete proofs, several relations between *analyticity and algebraicity*. First let M be a finite-dimensional vector space: all analytic subsets of the projective space $\mathbb{P}(M)$ are algebraic (Chow's theorem); an analytic set S in M, of constant dimension k, is algebraic if and only if there exist subspaces X, Y and positive constants c, α such that $M = X \oplus Y$, $\dim X = k$, and $x \in X$, $y \in Y$, $x + y \in$

$S \Rightarrow \|y\| \leq c(1 + \|x\|^\alpha)$ (Rudin's theorem); Hilbert's Nullstellensatz also exists for polynomials on M.

Bezout's very old theorem (his "Théorie générale des équations algébriques," which appeared in 1779) on the number of intersection lines, with multiplicities taken into account, of n algebraic cones in an $(n+1)$-dimensional vector space, now gets flawless statement and proof. By the Siegel-Thimm theorem, n functions f_j meromorphic on a compact manifold are algebraically dependent (i.e., $P(f_1, \ldots, f_n) = 0$ for some nonzero polynomial P) if and only if they are analytically dependent (i.e., the map $(f_1, \ldots, f_n)$ has rank $< n$ everywhere). Algebraic spaces are defined and the above quoted Chow's theorem extended to compact algebraic spaces. The end of the chapter is a detailed study of Grassmann manifolds—their embedding into projective spaces—and Chow's theorem on biholomorphic mappings of Grassmann manifolds.

When a review cannot be exhaustive, as is the case here, it can at least win readers for the book: this one deserves them, in all respects.

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