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IS THE BOUNDARY OF A SIEGEL DISK A JORDAN CURVE?

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Abstract. Bounded irreducible local Siegel disks include classical Siegel disks of polynomials, bounded irreducible Siegel disks of rational and entire functions, and the examples of Herman and Moeckel. We show that there are only two possibilities for the structure of the boundary of such a disk: either the boundary admits a nice decomposition onto a circle, or it is an indecomposable continuum.

1. Introduction

Let $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a rational map of the Riemann sphere of degree at least two. The dynamics of f divides $\overline{\mathbf{C}}$ into two disjoint sets: the *stable set* or *Fatou set*, and the *unstable set* or *Julia set*. On the Fatou set the dynamics of f is well behaved, while the dynamics of f on the Julia set is chaotic.

The work of Sullivan [Su] completed the understanding of the dynamics of f on the Fatou set. Every component of the Fatou set is eventually periodic, and essentially five kinds of dynamical behavior are possible on these domains. One of these behaviors is a Siegel disk.

A component G of the Fatou set of f is a *Siegel disk* if $f(G) = G$, G contains a neutral fixed point w_0, and $f\big|G$ is analytically conjugate to a rotation. Siegel [S] showed in 1942 that such disks exist.

To say that w_0 is a *neutral fixed point* means $f(w_0) = w_0$ and $|f'(w_0)| = 1$. Hence $f'(w_0) = e^{2\pi i\theta}$ for some real number θ in $[0,1)$. It is known that θ must be irrational, and much has been written in the effort to decide which irrationals yield a Siegel disk (see [B] or [M]).

The dynamics of the Julia set is more subtle, and much is still unknown. Douady and Sullivan [D1] have raised a very natural question: Is the boundary of a Siegel disk a Jordan curve? Herman [D2] has obtained an affirmative answer in special circumstances, but, in general, no answer is known.

More generally, let us define a *bounded local Siegel disk* to be a pair (G, F_θ), where G is a bounded simply connected domain in $\mathbf{C}$, and $F_\theta\colon G \to G$ is a conformal map complex analytically conjugate to a rotation through the irrational angle θ such that F_θ extends continuously to the boundary of G. The fixed point w_0 is again called a *Siegel point*. A bounded Siegel disk (G, F_θ) is *irreducible* if the boundary of G separates the Siegel point w_0 from ∞, but no proper closed subset of the

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boundary has this property. Bounded irreducible local Siegel disks include classical Siegel disks of polynomials as well as bounded irreducible Siegel disks of rational and entire functions and even the exotic examples of Herman [H1] and Moeckel [Mo].

We describe the structure of the boundaries of such domains by proving the following theorem.

Theorem 1.1. *The boundary ∂G of a bounded irreducible local Siegel disk satisfies exactly one of the following properties*:

1. *The inverse φ^{-1} of the Riemann map $\varphi\colon \mathbf{D} \to G$ of the conjugation extends continuously to a map $\psi\colon \partial G \to \partial \mathbf{D} = S^1$, or*
2. *∂G is an indecomposable continuum.*

An *indecomposable continuum* is a compact connected space which cannot be written as a union $A \cup B$ with A and B connected closed proper subsets of X. Indecomposable continua are complicated spaces; nevertheless, Herman [H1] has constructed a bounded irreducible local Siegel disk whose boundary is a certain indecomposable continuum known as the pseudocircle.

In case (1), the point inverses of $\psi\colon \partial G \to S^1$ are the impressions of prime ends of φ. In particular, the point inverses of ψ are connected. A space with such a decomposition onto a circle can be written as a union $A \cup B$ as described above, so the two possibilities are mutually exclusive. Moeckel [Mo] has constructed such an example in which the point inverses of ψ are either points or straight line intervals.

The Moeckel example shows that we cannot require φ^{-1} to extend to a homeomorphism in (1), while the Herman example shows that (2) can occur. Thus the result is the best possible for such local Siegel disks.

The boundary of a Siegel disk is a Jordan curve if and only if the Riemann map $\varphi\colon \mathbf{D} \to G$ of the conjugation extends to a homeomorphism of $\overline{\mathbf{D}}$ onto $\overline{G}$. This is equivalent, of course, to $\varphi^{-1}\colon G \to \mathbf{D}$ extending to a homeomorphism of $\overline{G}$ onto $\overline{\mathbf{D}}$. Thus we may interpret the theorem to imply any counterexample must be as nice as possible or as complicated as possible.

The theorem above implies that a weak additional hypothesis is enough to answer the Douady-Sullivan question affirmatively.

Theorem 1.2. *Let A be an arc in the boundary of a Siegel disk of a polynomial of degree $d \geq 2$. If two internal rays from G land on A, then ∂G is a Jordan curve.*

Thus, any arc in a counterexample must be "hidden." In particular, we have the following corollary.

Corollary 1.3. *If the boundary of a Siegel disk of a polynomial of degree $d \geq 2$ is arcwise connected, then ∂G is a Jordan curve.*

The Julia set of a polynomial f is the closure of the set of repelling periodic points of f, and the boundary of a Siegel disk is a subset of the Julia set. Hence the next theorem is in one sense a little surprising.

Theorem 1.4. *If the boundary ∂G is a Siegel disk of a polynomial of degree $d \geq 2$ contains a periodic point, then ∂G is an indecomposable continuum.*

This paper is an abstract of the results in [R3]. The paper [R3] contains a brief history of indecomposable continua occurring in the study of dynamical systems and suggests that it is not so unexpected that we must deal with indecomposable continua in this situation.

2. The structure of the boundary of local Siegel disks

Let (G, F_θ) denote a bounded irreducible local Siegel disk. We need a number of tools to complete the proof of the structure theorem.

The first is a result of Pommerenke and Rodin [PR] about prime ends η and their impressions $I(\eta)$.

Theorem 2.1. *Each local Siegel disk has a Pommerenke-Rodin number; i.e., there exists a number* d (*not to be confused with the degree of a polynomial*) *with* $0 \leq d \leq 2$ *such that, for prime ends* η_1 *and* η_2 *in* $\partial\mathbf{D}$,

$$I(\eta_1) \cap I(\eta_2) \neq \emptyset \Leftrightarrow |\eta_1 - \eta_2| \leq d\,.$$

The distance $|\eta_1 - \eta_2|$ on $\partial\mathbf{D}$ is given by the Euclidean metric on $\mathbf{C}$; hence, for example, $|\eta_1 - \eta_2| = 2$ if and only if η_1 and η_2 are diametrically opposite. It follows that $d = 0$ if and only if all impressions are pairwise disjoint, while $d = 2$ if and only if each pair of impressions has a point in common.

The second and most important tool is the theory of prime ends as related to indecomposable continua. The work of Rutt [Ru] is used, for instance, in proving the following result of the author [R1], a result that enables us to recognize indecomposable continua by analytic methods.

Theorem 2.2. *If* $(G, \mathcal{F}_\theta)$ *is a local Siegel disk, then* ∂G *is an indecomposable continuum if and only if there exists a prime end* η *of* G *such that the impression* $I(\eta) = \partial G$.

The proof of the structure theorem is completed by a somewhat delicate analysis of the relationship between prime ends and indecomposable continua. The details appear in [R3].

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