

VOICULESCU THEOREM, SOBOLEV LEMMA, AND EXTENSIONS OF SMOOTH ALGEBRAS

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Dedicated to the memory of Xian-Rong Wang

ABSTRACT. We present the analytic foundation of a unified B-D-F extension functor Ext_τ on the category of noncommutative smooth algebras, for any Fréchet operator ideal \mathcal{K}_τ . Combining the techniques devised by Arveson and Voiculescu, we generalize Voiculescu's theorem to smooth algebras and Fréchet operator ideals. A key notion involved is τ -smoothness, which is verified for the algebras of smooth functions, via a noncommutative Sobolev lemma. The groups Ext_τ are computed for many examples.

1. INTRODUCTION

For a compact manifold M , the extensions of $C(M)$ by the compact operators $\mathcal{K}(\mathcal{H})$ form an abelian group $\text{Ext}(M)$, coinciding with the odd K -homology $K^1(M)$ [BDF], which can also be defined in terms of the elliptic operators [A1]. This was a starting point of [K].

For a Schatten ideal \mathcal{L}^p , the notion of \mathcal{L}^p -smooth elements in $\text{Ext}(M)$ was introduced and studied in [D, DV], and generalized in [S1, G]. Connes constructed the Chern character of extensions of smooth algebras by \mathcal{L}^p in the cyclic cohomology of the smooth algebras [C]. When $p = 1$ it recovers the trace forms in [HH, CP].

Today Ext -theory for C^* -algebras has developed into a multifaced field of fundamental importance in modern analysis, as the meeting ground of classical operator theory, in particular Toeplitz operators, Wiener-Hopf operators, and noncommutative differential geometry [C], especially pseudodifferential operators, index theory, K -theory, and cyclic homology.

While our knowledge of topological Ext -theory is rather complete, a natural and fundamental problem [A2, p. 9; D, p. 68; HH, p. 236] remains wide open in noncommutative differential geometry, i.e., the formulation and understanding of the extension theories of smooth algebras. It would naturally serve as the domain of the Chern character defined in [C]. Since smooth algebras are Fréchet algebras, it is desirable to have the extension theory constructed for Fréchet operator ideals.

In this note we present the analytical foundation of such a general theory, which produces a functor Ext_τ from the category of smooth algebras to abelian groups,

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for any Fréchet operator ideal \mathcal{K}_τ [P; W1, 10.11]. Such a theory will establish a unified framework into which all previous results in this direction fit.

There are two well-known technical results forming the cornerstone of Brown-Douglas-Fillmore theory. One is the celebrated Voiculescu's theorem [V1], which generalizes Weyl-von Neumann-Berg theorem to C^* -algebras. In [V2] it is extended to normed ideals and algebras with countable bases. The other is Stinespring's theorem [S]. We generalize both theorems to Fréchet ideals and smooth algebras, answering a question in [D], and we illustrate the theory in the case of smooth manifolds.

2. SMOOTH CATEGORY AND VOICULESCU'S THEOREM

By a *smooth algebra* we mean a Fréchet $*$ -algebra A^∞ equipped with a norm p such that $p(a^*a) = p(a)^2$ for all $a \in A^\infty$. Often it is denoted as a pair (A^∞, A) , where A is the C^* -completion of A^∞ with respect to p . A prototype is $(C^\infty(M), C(M))$ for a compact smooth manifold.

The *smooth category* is the category of all separable smooth algebras, with morphisms given by $*$ -homomorphisms of Fréchet $*$ -algebras while contractive with respect to the C^* -norms.

There are two key ingredients in our analysis. Both can be taken for granted for C^* -algebras. One is the existence of quasi-central approximate identity, identified in [Ar, V2]. The other is τ -smoothness, which had not appeared in literature until now.

We say a completely positive map $\phi: A^\infty \rightarrow \mathcal{L}(\mathcal{H})$ is τ -quasi-central, if there is an increasing sequence $\{k_m\}$ of finite rank positive operators strongly converging to 1 such that

$$\lim_{m \rightarrow \infty} [k_m, \phi(a)] = 0 \quad \text{in } \mathcal{K}_\tau \text{ for every } a \in A^\infty.$$

Let (A^∞, A) be a smooth algebra with a dense $*$ -subalgebra A_0^∞ , which is countably generated as a vector space. A completely positive map ϕ from (A^∞, A) into $\mathcal{L}(H)$ is τ -smooth (or \mathcal{K}_τ -smooth) with respect to A_0^∞ , if the following property holds:

If ϕ_0 is a completely positive map from A into $\mathcal{L}(H)$ such that

$$(1) \quad (\phi - \phi_0)(A_0^\infty) \subset \mathcal{K}_\tau(H),$$

then

$$(\phi - \phi_0)(A^\infty) \subset \mathcal{K}_\tau(H).$$

For $(A^\infty, A) \subset \mathcal{L}(\mathcal{H})$, if we add $(\phi - \phi_0)(A^\infty \cap \mathcal{K}_\tau(\mathcal{H})) \subset \mathcal{K}_\tau(H)$ in (1), then we obtain the definition of a τ -smooth mod \mathcal{K}_τ (or \mathcal{K}_τ -smooth mod \mathcal{K}_τ) map.

A smooth operator algebra (A^∞, A) is called τ -smooth (or τ -smooth mod \mathcal{K}_τ) if the map id_{A^∞} is so. The following is the generalized Voiculescu's theorem.

Theorem 1. *Let (A^∞, A) be a separable operator algebra τ -smooth mod \mathcal{K}_τ in $\mathcal{L}(\mathcal{H})$. Let π be a nondegenerate τ -quasi-central representation of (A^∞, A) into $\mathcal{L}(\mathcal{H})$ such that $\pi|_{A \cap \mathcal{K}(\mathcal{H})} = 0$. Then there are unitaries $U_n: \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$ such that*

$$(U_n^*(a \oplus \pi(a))U_n - a) \in \mathcal{K}_\tau(\mathcal{H}) \quad \text{for every } a \in A^\infty,$$

$$\lim(U_n^*(a \oplus \pi(a))U_n - a) = 0 \quad \text{in } \mathcal{K}_\tau(\mathcal{H}) \text{ for every } a \in A_0^\infty.$$

Restricting the ideal \mathcal{K}_τ and A^∞ to various categories, one recovers all the previous results in this direction. The idea of the proof is an extension of the techniques invented in [Ar] and [V2] for Fréchet algebras.

3. SMOOTH EXTENSIONS

A τ -smooth extension of (A^∞, A) is a pair (π, P) , where π is a representation of A in a Hilbert space \mathcal{H} and P is a projection in \mathcal{H} such that (i) $[\pi(a), P] \in \mathcal{K}_\tau(\mathcal{H})$ for $a \in A^\infty$, and (ii) $P\pi(A)P \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Define a completely positive map $\phi(a) := P\pi(a)P$, $a \in A$. Then $E^\infty := \phi(A^\infty) + \mathcal{K}_\tau$ is a Fréchet *-algebra with the locally convex final topology induced by the maps $i: \mathcal{K}_\tau \rightarrow E^\infty$ and ϕ , and (E^∞, E) is a smooth operator algebra with a dense *-subalgebra $E_0^\infty := \phi(A_0^\infty) + \mathcal{K}_f$ countably generated as a vector space.

Two τ -smooth extensions are unitarily equivalent if the two associated completely positive maps on A^∞ are unitarily conjugate up to \mathcal{K}_τ -compact perturbation. A τ -smooth extension is degenerated if it is unitarily equivalent to a representation.

Let $\mathcal{E}xt_\tau(A^\infty)$ be the unitary equivalence classes of the τ -smooth extensions of A^∞ . A spatial isomorphism $M_2(\mathcal{L}(\mathcal{H})) \simeq \mathcal{L}(\mathcal{H})$ turns $\mathcal{E}xt_\tau(A^\infty)$ into an abelian semigroup. The quotient abelian monoid modulo the degenerate extensions will be denoted by $\text{Ext}_\tau(A^\infty)$.

We shall denote by $\mathcal{E}xt_{s,\tau}(A^\infty)$ those represented by τ -smooth completely positive maps. Replacing $\mathcal{E}xt_\tau$ by $\mathcal{E}xt_{s,\tau}$ in the above, we get the submonoid $\text{Ext}_{s,\tau}(A^\infty)$ of those consisting of τ -smooth completely positive maps. By a refinement of Stinespring's theorem [S], we can show that Ext_τ is a contravariant functor from the category of smooth algebras to the category of abelian groups.

If τ is finer than τ' , there is a natural transformation $\alpha_{\tau,\tau'}$ from Ext_τ to $\text{Ext}_{\tau'}$. In particular, there is a natural transformation α_τ from Ext_τ to the B-D-F functor Ext for any τ . Theorem 1 implies

Corollary. *If every extension in $\mathcal{E}xt_\tau(A^\infty)$ can be represented by a τ -smooth completely positive map and is τ -quasi-central if it is degenerate, then there is a natural isomorphism of abelian groups*

$$\text{Ext}_\tau(A^\infty) \simeq \mathcal{E}xt_\tau(A^\infty). \quad \square$$

We check the two conditions above for the case of commutative smooth algebras (A^∞, A) where $A = C(M)$, for a compact smooth manifold M of dimension n . We may assume M is embedded in \mathbb{R}^N ($N \leq 2n$, by Whitney's theorem). From the deep results in [V2], we have

Theorem 2 (Voiculescu). *Notation is as above. All degenerated smooth extensions of (A^∞, A) by \mathcal{L}^N are \mathcal{L}^N -quasi-central.*

For the other condition we have

Theorem 3. *Let M be a compact $C^{(k)}$ -manifold of dimension n . For any Fréchet operator ideal \mathcal{K}_τ , any completely positive map defining extensions of $C^{(n+2+\varepsilon)}(M)$, $C(M)$ by \mathcal{K}_τ is always τ -smooth, for any $\varepsilon > 0$. Here we assume $k \geq n + 2 + \varepsilon$.*

The key step in the proof is the following noncommutative Sobolev lemma. It implies that if a sequence (a_{m_1, \dots, m_n}) belongs to a Sobolev space H_s for sufficiently large s , then the quantized generalized function determined by (a_{m_1, \dots, m_n}) converges in the τ -smooth quantized Fréchet algebra.

Lemma. *Let (A^∞, A) be a commutative smooth algebra with n generators $\{x_1, \dots, x_n\}$, such that*

- (1) *The C^* -norm $\|x_j\| \leq 1$, for all $j = 1, \dots, n$;*
- (2) *Any element in A^∞ has the form*

$$f = \sum_{(m_1, \dots, m_n) \in \mathbb{Z}^n} a_{m_1, \dots, m_n} x_1^{m_1} \dots x_n^{m_n},$$

such that

$$\sum_{(m_1, \dots, m_n) \in \mathbb{Z}^n} a_{m_1, \dots, m_n} (|m_1| + \dots + |m_n|)^2 < \infty.$$

For any Fréchet operator ideal \mathcal{K}_τ , let ϕ be the completely positive map associated to any τ -smooth extension of (A^∞, A) . Then ϕ is τ -smooth.

4. EXAMPLES

1. Let $A = C(X)$ where X is a compact, second countable, totally disconnected space. Then A has a single generator. If A^∞ also does, then $\text{Ext}_{s, \tau}(A^\infty) = 0$.

2. Let $A = A^\infty = C(S^1)$. There is a representation $\pi: A \rightarrow \mathcal{L}(H)$ defining a degenerate τ -smooth extension, which is \mathcal{K} -smooth but not \mathcal{L}^1 -smooth as a completely positive map. Thus $\text{Ext}_{\mathcal{L}^1}$ is not a compatible functor for the category of C^* -algebras.

3. For any $p > 1$, there is a degenerate faithful representation of $(L^1(T), C^*(T))$ defining a \mathcal{L}^p -smooth extension, which is not \mathcal{L}^p -smooth as a completely positive map.

This shows that if $A = C^*(G)$ for a compact Lie group G , $L^1(G)$ is too large a smooth subalgebra. One needs to take, for example $C^\infty(G)$ instead.

4. For any operator ideal \mathcal{K}_τ , we have

$$\text{Ext}_\tau C^\infty(S^1) \simeq \text{Ext} C(S^1) \simeq \mathbb{Z}.$$

Fix $\mathcal{K}_\tau \subset \mathcal{L}^p$ for some $p \geq 1$. It follows from [C] that there is a natural group homomorphism

$$ch_m: \text{Ext}_\tau(A^\infty) \rightarrow HC^{2m+1}(A^\infty), \quad m \geq p,$$

such that $ch_{m+1} = S \circ ch_m$. Here HC^* is the cyclic homology.

5. Let D be the unit disc. Then $\text{Ext}(C(D)) = 0$. However, $\text{Ext}_{\mathcal{L}^1} C^\infty(D) \otimes \mathbb{C}$ contains as a direct summand the group $HC^1(C^\infty(D))$, which is the space of all the closed de Rham currents on D of dimension 1.

Remarks. Since there is no hausdorff topology on $\mathcal{L}/\mathcal{K}_\tau$, we abandon the conventional formulation of Ext ; so a lifting problem does not arise.

An attractive perspective of the smooth extension theory is that $\text{Ext}_\tau(A^\infty)$ is a differential invariant for appropriate \mathcal{K}_τ and A^∞ . In [Km] it is shown that even in the B-D-F group the class of a smooth extension may depend on the smooth structure.

Details will appear elsewhere (see [W2, W3, W5]). We plan to investigate the topological aspect of the theory, along with the even degree functors in a future work.

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REFERENCES

- [A1] M. F. Atiyah, *Global theory of elliptic operators*, Proc. Int. Conf. Funct. Anal. and Related Topics, Univ. Tokyo Press, Tokyo, 1970, pp. 21–23.
- [A2] ———, *A survey of K-theory*, Lecture Notes in Math., vol. 575, Springer-Verlag, New York, 1977, pp. 1–9.
- [Ar] W. Arveson, *Notes on extensions of C^* -algebras*, Duke Math J. **44** (1977), 329–355.
- [B] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Math. **38** (1974), 1109–1165.
- [Bg] I. D. Berg, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971), 365–371.
- [B-C] C. A. Berger and L. A. Coburn, *Toeplitz operators and quantum mechanics*, J. Funct. Anal. **68** (1986), 273–299.
- [BC] B. Blackadar and J. Cuntz, *Differential Banach algebra norms and smooth subalgebras of C^* -algebras*, preprint.
- [BDF] L. Brown, R. Douglas, and P. Fillmore, *Extensions of C^* -algebras and K-homology*, Ann. of Math. (2) **105** (1977), 265–324.
- [CP] R. Carey and J. D. Pincus, *Almost commuting algebras, K-theory and operator algebras*, Lecture Notes in Math. vol. 575, Springer-Verlag, New York, 1977, pp. 19–43.
- [C] A. Connes, *Noncommutative differential geometry*. I, II, Inst. Hautes Études Sci. Publ. Math. **62** (1986), 257–360.
- [CK] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, C. R. Acad. Sci. Paris Sér. I. Math. **299** (1984), 963–968.
- [D] R. Douglas, *On the smoothness of elements of Ext*, Topics in modern operator theory, Birkhauser, 1981, pp. 63–69.
- [DV] R. Douglas and D. Voiculescu, *On the smoothness of sphere extensions*, J. Operator Theory **6** (1981), 103–111.
- [E] E. Effros, *Advances in quantized functional calculus*, Proc. I.C.M. (1986), 906–916.
- [G] G. Gong, *Smooth extensions for a finite CW-complex*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 73–78.
- [HH] J. W. Helton and R. Howe, *Integral Operators: commutators, traces, index, and homology*, Lecture Notes in Math. vol. 345, Springer-Verlag, New York, 1973, pp., 141–209.
- [Km] J. Kaminker, *Pseudo-differential operators and differential structures*. II, Amer. J. Math. (1986), 703–718.
- [K] G. Kasparov, *The operator K-functor and extensions of C^* -algebras*, Izv. Akad. Nauk SSSR Ser. Math. **44** (1980), 571–636.
- [P] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1980.
- [Sl] N. Salinas, *Smooth extensions and smooth joint quasitriangularity*, Oper. Theory: Adv. Appl. vol. 11, Birkhauser Baseland, Boston, 1983, pp., 303–332.
- [S] W. F. Stinespring, *Positive functions on C^* -algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.
- [V1] D. Voiculescu, *A noncommutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), 97–113.
- [V2] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators*, J. Operator Theory **2** (1979), 30–37; II **5** (1981), 77–100.
- [W1] X. Wang, *KK-theories for topological algebras*, K-theory **5** (1991), 97–150.
- [W2] X. Wang, *Theorems of Voiculescu and Stinespring; Extensions of smooth algebras* (to appear).
- [W3] X. Wang, *Smooth extensions and quantized Fréchet algebras dimensions 0 and 1* (to appear).
- [W4] X. Wang, *Quantizations, quantized smooth manifolds and invariants*, Proc. NATO conference (to appear).

- [W5] X. Wang, *Smooth K-homology, Chern character, and a noncommutative Sobolev Lemma*, in preparation.
- [W6] X. Wang, *Geometric BRST quantization and smooth Toeplitz extensions*, in preparation.

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