7. F. Dumortier, R. Roussarie, and C. Rousseau, *Hilbert's 16th problem for quadratic vector fields,* preprint, 1992.

8. J. Écalle, J. Martinet, R. Moussu, and J-P. Ramis, *Non-accumulation des cycles-limites* I, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 375–377.

9. J. Écalle, *Finitude des cycles limites it accéléro-sommation de l'application de retour,* Lecture Notes in Math., vol. 1455, Springer-Verlag, Berlin and New York, 1990, pp. 74–159.

10. D. Hilbert, *Mathematical problems,* (M. Newton, transl.), Bull. Amer. Math. Soc **8** (1902), 437–479.

11. Yu. S. Il'yashenko, *Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane,* Funct. Anal. Appl. **18** (1984), 199–209.

12. ———, *"On limit cycles" and related problems of the local theory of differential equations,* Russian Math. Surveys **40** (1985), 1–49.

13. ———, *The finiteness theorem for limit cycles,* Uspekhi Mat. Nauk. **42** (1987), 223. (Russian)

14. ———, *Finiteness theorems for limit cycles,* Russian Math. Surveys **45** (1990), 129–203.

15. H. Poincaré, *Mémore sur les courbes définies par une equation différentielle,* J. Mathématiques 7 (1881), 375–422.

16. J. Reyn, *A bibliography of the qualitative theory of quadratic systems of differential equations in the plane,* TU Delft Technical Report 89-71, 1989.

17. S-L. Shi, *A concrete example of the existence of four limit cycles for plane quadratic systems,* Sci. Sinica **23** (1980), 153–158.

18. J. Sotomayor and R. Paterlini, *Quadratic vector fields with finitely many periodic orbits,* Lecture Notes in Math., vol 1007, Springer-Verlag, Berlin and New York, 1983, pp. 753–766.

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Quantum physics, relativity and complex spacetime, by Gerald Kaiser. North-Holland Mathematics Studies, vol. 163, North-Holland, Amsterdam, xvi + 359 pp., $85.75. ISBN 0-444-88465-3

The history of science, technology, and mathematics suggests that it is fruitful to consider how various areas of mathematics and physics are linked to each other. As examples, Newtonian Mechanics, Quantum Mechanics, and Einstein Gravitation Theory are tied to the Theory of Ordinary Differential Equations, Hilbert Space Theory, and Riemannian Geometry, respectively. The same sort of unified overview can be applied to various parts of Engineering: Computer Science is tied to Logic and the Algebraic Theory of Languages, and Control Theory to the Lie Theory of Vector Field Systems and the Theory of Stochastic Processes.

In this very interesting book, the author investigates the interrelation between various Lie and complex function theories on the mathematical side and quantum mechanical and field-theoretic theories on the physics side. Many of the situations he considers involve the theory of deformation of Lie groups. His point of view, however, is very much that of the 'down-to-earth' mathematical physicist and there is very little said about the general setting for the work. The benefit to the reader is that, rather than wading through a morass of definitions and explanations of general concepts, one can (hopefully) learn on-the-read; but, I must say that an appendix on the mathematical background would have been very useful!

What geometers call the 'theory of complex analytic structures' has long played an informal role in physics, since analyticity questions have arisen ever since the nineteenth century. The classical 1-D complex function theory has always appealed to mathematical physicists because not only does it fit in well with their computational view of the world, but it also has great mathematical simplicity and beauty. The explicit solutions of such linear PDEs as the Wave and Maxwell Equations—especially their Fundamental Solutions, which the physicists call 'Green's Functions' or 'Propagators'—also involve functions with fascinating analyticity properties. The attempts that were made in the 1950s and 1960s to understand quantum field theory from an 'axiomatic' point of view also introduced such analytic structures.

Similarly, Lie groups have been basic mathematical structures for physics ever since the work of Weyl in the 1920s, although until the past twenty-five years they were—as mathematical structures—regarded by most physicists with the same affection that locusts inspire in farmers' hearts. This residual mistrust of general concepts is, perhaps, reflected in the fact that there is no entry called 'Lie Group' in the index, nor could I find a definition of the concept. The theory of deformations of Lie groups is even less well known by mathematical physicists (or understood, for that matter) and there is no indication here that there even is such a theory. The author talks informally about a concept called 'contraction', which may be regarded as one special type of deformation.

Ever since the quantum mechanical foundational work of von Neumann, Weyl, and Wigner, it has been evident that a basic mathematical structure for quantum mechanics consists of the following data:

(a) a Lie group G,
(b) a Hilbert space H,
(c) a linear representation ρ of G by linear transformations on H.

Given the relation between the norm of vectors in H and 'probabilities' of quantum events, it is natural to assume that the transformation group action of G on H defined by ρ preserves the norm on H associated with the Hilbert space structure. Given natural hypotheses about the continuity properties of ρ, this assumption determines certain relations between the topological structure of G and the algebraic and analytic properties of ρ. For example, ρ restricted to the identity component of the identity of G is a unitary representation of G, and the representation that ρ induces on the Lie algebra of G consists of operators that are skew-adjoint operators. In many physical situations the triple (G, H, ρ) depends 'on parameters', (e.g., 'the velocity of light', 'mass of particles', 'Planck's constant'), which leads to a deformation theory.

Often these structures have a geometric genesis. G acts as a group of linear automorphisms of a vector bundle. H, as a vector space, consists of a set of cross-sections, and the representation ρ is that arising from the natural geometric action on cross-sections. This brings the study of the underlying mathematics into the domain of what is sometimes called 'geometric analysis'.

Now, to each such Lie group G one can define a *complexification* as any complex analytic Lie group G' whose Lie algebra is isomorphic to the complexification of the Lie algebra of G. If G is simply connected, there is up to isomorphism just one such simply connected G'. If (G, H, ρ) is the structure defined above, and in addition G is simply connected and H is finite dimensional, then the 'matrix elements' of ρ, considered as complex number-valued functions on G, can be extended globally to be complex analytic functions on G'. However, if H is infinite dimensional—as it will usually be in the situations encountered in physics—this extension property will break down globally, although such extensions will often exist locally. This leads to an interesting analyticity structure, which was studied in the nineteenth century using traditional complex analysis technique for such 'matrix element functions' as the Bessel and Legendre functions. Such questions are frequently encountered in work on the mathematical foundations of quantum field theory.

After these general remarks, I turn to Kaiser's book. It is very personal, dealing mainly with areas in mathematical physics that have interested him since his student days. Luckily, much of this is now very timely, and Kaiser has many interesting things to say about such topics as wavelets, coherent states, and the relation between the group-theoretic and complex-analytic structure of quantum fields and associated partial differential equations. A central concept is that of a *generalized frame structure* for a Hilbert space associated with a Lie group representation. This is a structure that, on the one hand, generalizes the notion of 'basis of a vector space' in linear algebra and 'resolution of the identity' in functional analysis and, on the other, is related to the theory of induced representations and reproducing kernels of Hilbert spaces.

The core of the book is Chapters Four and Five, titled "Complex Spacetime" and "Quantized Fields". They mainly deal with the Galilean and Poincaré groups and their complexifications, deformations, and contractions, considered in terms of transformation groups and geometric structures on R^4 and C^4. The author discusses the relation between these transformation groups, various linear representations of these groups, frames, reproducing kernels, coherent states, and quantum fields. One of his main points is that some aspects of the quantum mechanics of relativistic particles and fields are closely linked to the complexified space-time manifold. There is much here that should be of great interest to both mathematicians interested in physical motivation and physicists trying to understand the Lie-theoretic and complex-analytic fundamentals of their discipline.

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