

BOOK REVIEW

Spectral and scattering theory for wave propagation in perturbed stratified media,
by Ricardo Weder. Appl. Math. Sci., vol. 87, Springer-Verlag, Berlin and New
York, 188 + pp., \$35.00. ISBN 0-387-97357-5

INTRODUCTION

This monograph is volume 87 in the Springer series entitled *Applied Mathematical Sciences*. The series is an eclectic one with volumes on many topics in pure and applied mathematics. This volume is a contribution to modern mathematical physics as defined by the four volume work of M. Reed and B. Simon [5–8]. The monograph presents applications of abstract functional analysis to two scattering problems of classical physics. The first treats the scattering of acoustic (scalar) waves by localized inhomogeneities and/or obstacles in a plane stratified fluid. The second treats the analogous problem for electromagnetic (vector) waves.

The method used in the monograph has a history going back to work of A. Y. Povzner [4] in 1955 and T. Ikebe [1] in 1960 on quantum mechanical potential scattering. Subsequently many scattering problems of classical and quantum physics were studied by this method. They include scattering of acoustic and electromagnetic waves by localized inhomogeneities and/or obstacles in homogeneous media and in cylindrical waveguides, scattering by diffraction gratings and other periodic structures, and many others. A review of the literature up to the late 1970s may be found in Reed and Simon III [7]. Recently this kind of work has gone out of fashion, although many difficult and interesting problems remain unsolved. An exception is the study of inverse scattering problems, which has been a very active field in the last decade.

The results developed in the monograph can be described in the formalism of the abstract theory of scattering with two Hilbert spaces [3]. There one has a pair of Hilbert spaces, \mathcal{H}_0 and \mathcal{H} , together with corresponding selfadjoint operators, A_0 and A , and a bounded linear bijection $J: \mathcal{H}_0 \rightarrow \mathcal{H}$. The principal objects of study are the wave operators W_+ and W_- and the scattering operator $S = W_+^* W_-$. The wave operators are formally defined by

$$(1) \quad W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itA} J e^{-itA_0} P_0,$$

where P_0 is the orthogonal projection of \mathcal{H}_0 onto the subspace of absolute continuity of A_0 and s -lim denotes the strong limit. It is known that, when they exist, W_+ and W_- are partial isometries [2] with initial set $\mathcal{H}_{0,ac} = P_0 \mathcal{H}_0$ and final set contained in the analogous subspace \mathcal{H}_{ac} for A . W_+ and W_- are said to be complete if their final sets are precisely \mathcal{H}_{ac} . If W_+ and W_- are complete then S is unitary in $\mathcal{H}_{0,ac}$ [2].

DESCRIPTION OF THE WORK

Following a brief (two page) introduction, the monograph consists of two major chapters (Chapters 2 and 3) plus two technical appendices and a two page discussion of related literature. Chapter 2 develops spectral and scattering theory for acoustic waves in inhomogeneous fluids that are local perturbations of plane stratified fluids. Chapter 3 develops a parallel theory for electromagnetic waves. Here, for brevity, only the acoustic case will be discussed.

The monograph treats acoustic wave propagation in an inhomogeneous fluid that is governed by the wave equation

$$(2) \quad u_{tt}(x, y, t) - c^2(x, y)\Delta u(x, y, t) = 0,$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, $t \in \mathbf{R}$, and Δ is the $(n + 1)$ -dimensional Laplacian in the coordinates (x, y) . The function $c(x, y)$ is interpreted as the sound speed at position (x, y) . Equations of the form (2) are studied that are perturbations of plane stratified fluids, governed by the equation

$$(3) \quad u_{tt}(x, y, t) - c_0^2(y)\Delta u(x, y, t) = 0,$$

with sound speed $c_0(y)$ at depth y . It is assumed that $c(x, y)$ and $c_0(y)$ have bounds $0 < c_m \leq c(x, y)$, $c_0(y) \leq c_M < \infty$. Moreover, to lighten the technical difficulties, the proofs are limited to the case of the Pekeris profile

$$c_0(y) = \begin{cases} c_+ & \text{for } y \geq h, \\ c_h & \text{for } 0 \leq y < h, \\ c_- & \text{for } y < 0, \end{cases}$$

where $c_h < c_+ \leq c_-$. This restriction is removed in the treatment of the electromagnetic case.

Spectral and scattering theory are developed for equations (2) and (3) under the assumption that $c(x, y)$ and $c_0(y)$ are close at infinity in the sense that for all $(x, y) \in \mathbf{R}^{n+1}$ one has

$$|c(x, y) - c_0(y)| \leq C(1 + |x| + |y|)^{-\alpha}, \quad \alpha > 1.$$

The formalism of the abstract theory of scattering is introduced by defining the operator A_0 to be a selfadjoint realization of the operator

$$A_0 = -c_0^2(y)\Delta$$

in the Hilbert space \mathcal{H}_0 with scalar product

$$(u, v)_{\mathcal{H}_0} = \int_{\mathbf{R}^{n+1}} u(x, y) \overline{v(x, y)} c^{-2}(x, y)_0(y) dx dy,$$

while the operator A is a selfadjoint realization of the operator

$$A = -c^2(x, y)\Delta$$

in the Hilbert space \mathcal{H} with scalar product

$$(u, v)_{\mathcal{H}} = \int_{\mathbf{R}^{n+1}} u(x, y) \overline{v(x, y)} c^{-2}(x, y) dx dy.$$

The pair (\mathcal{H}, A) is appropriate for the Cauchy problem for the wave equation (2). The monograph also treats the initial-boundary value problem for (2) with (x, y) in an exterior domain Ω ($\mathbf{R}^{n+1} - \Omega$ bounded). In this case functions in the domain of A are required to satisfy a Dirichlet, Neumann, or Robin boundary condition.

The starting point for the research presented in this monograph is the reviewer's monograph [9]. There a complete spectral analysis and generalized eigenfunction expansion were developed for the operator A_0 . The spectral theorem and generalized eigenfunctions were used to construct a unitary operator

$$F: \mathcal{H}_0 \rightarrow \widehat{\mathcal{H}} = \bigoplus_{j=0}^{\infty} \widehat{\mathcal{H}}_j,$$

which was spectral for A_0 in the sense that $FA_0\varphi = \lambda F\varphi$ for all φ in the domain of A_0 . F is called the generalized Fourier map for A_0 .

The primary goal of the monograph is to prove the existence and completeness of the wave operators W_{\pm} for the pair A_0, A . The main steps of the proof may be described briefly as follows.

Step 1. The spectral theorem and the generalized Fourier map F are used to prove that the resolvent operator

$$R_0(z) = (A_0 - z)^{-1}$$

has boundary values $R_0(\lambda \pm i0)$ at the points

$$\lambda \in [0, \infty) = \text{spectrum of } A_0.$$

In scattering theory this is called the limiting absorption principle. The convergence of $R_0(z)$ is in the uniform operator topology of bounded operators between certain weighted Sobolev spaces (and not in \mathcal{H}_0).

Step 2. The limiting absorption principle for A_0 and a local compactness theorem for the resolvent of A ,

$$R(z) = (A - z)^{-1},$$

are used to prove a limiting absorption principle for A ; that is, the existence of the limits $R(\lambda \pm i0)$.

Step 3. The operators $R(\lambda \pm i0)$ and the spectral theorem are used to define generalized Fourier maps F_+ and F_- for A such that

$$F_{\pm}: \mathcal{H}_{ac} \rightarrow \widehat{\mathcal{H}},$$

and F_+ and F_- are unitary and spectral for A : $F_{\pm}A\varphi = \lambda F_{\pm}\varphi$.

Step 4. Using the preceding results, it is shown that W_+ and W_- are given by the generalized Fourier maps as

$$(4) \quad W_{\pm} = F_{\pm}^* F.$$

This representation and the properties of F and F_{\pm} described above imply that W_+ and W_- are complete.

The S Matrix. Equations (4) imply that the scattering operator $S = W_+^* W_-$ has the representation

$$S = F^* \widehat{S} F, \quad \text{where } \widehat{S} \text{ is defined by } \widehat{S} = F_+ F_-^* .$$

\widehat{S} is a unitary operator in $\widehat{\mathcal{H}}$. In the final portion of Chapter 2 it is shown that $\widehat{\mathcal{H}}$ has a direct integral structure

$$\widehat{\mathcal{H}} = \bigoplus \int_0^\infty \widehat{\mathcal{H}}(\lambda) d\lambda,$$

which reduces \widehat{S} so that

$$\widehat{S} = \bigoplus \int_0^\infty S(\lambda) d\lambda .$$

The existence of the operators $S(\lambda)$ is proved, via the limiting absorption principle for A , and it is shown that $S(\lambda) - I$ is compact and the mapping $\lambda \rightarrow S(\lambda)$ is locally Hölder continuous with Hölder exponent $\gamma < 1/2$. The chapter ends with some results on the continuous dependence of the wave and scattering operators on the perturbation.

Intended readers. The brief and highly simplified description above omits mention of many difficulties the author overcame in preparing this work. In fact, the work is intricate and fraught with technical difficulties throughout. Moreover, the monograph is written in a terse, fast-paced style and employs much of the functional analytic machinery of modern mathematical physics. Thus, in order to follow the exposition in detail one needs to have a working knowledge of a substantial portion of the Reed-Simon books [5–8] or equivalent material. Mathematical physicists and mathematicians with this background should find the work stimulating and useful.

The introduction to the monograph states that the work is also intended for applied mathematicians, physicists, and engineers. The usefulness of the work to these groups is doubtful. Applied mathematicians have traditionally disliked and shunned what they regard as the overly cumbersome methods of functional analysis. Also, the subject is not a part of most physics and engineering curricula. These readers will find the monograph very difficult to penetrate.

Concluding remarks. The monograph is largely based on the author's own research during the late 1980s. Many of the results in it are new or are proved for the first time: see the Notes. These include the results on the existence and Hölder continuity of the trace maps associated with the unperturbed acoustic propagator (Lemmas 2.1 and 2.2), and the results on the absence of positive eigenvalues for the perturbed acoustic propagator (Theorems 5.1 and 5.4). These results, and the techniques for proving them, should have applications to many other scattering problems, old and new.

REFERENCES

1. T. Ikebe, *Eigenfunction expansions associated with the Schrödinger operator and their application to scattering theory*, Arch. Rational Mech. Anal. **5** (1960), 1–34.
2. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin and New York, 1966.
3. ———, *Scattering theory with two Hilbert spaces*, J. Funct. Anal. **1** (1967), 342–369.

4. A. Y. Povzner, *On eigenfunction expansions in terms of scattering solutions*, Dokl. Akad. Nauk SSSR **104** (1955).
5. M. Reed and B. Simon, *Methods of Mathematical Physics*. I, Academic Press, New York, 1972.
6. ———, *Methods of Mathematical Physics*. II, Academic Press, New York, 1975.
7. ———, *Methods of Mathematical Physics*. III, Academic Press, New York, 1979.
8. ———, *Methods of Mathematical Physics*. IV, Academic Press, New York, 1978.
9. C. H. Wilcox, *Sound propagation in stratified fluids*, Appl. Math. Sci., vol. 50, Springer-Verlag, Berlin and New York, 1984.

CALVIN H. WILCOX
UNIVERSITY OF UTAH
E-mail address: wilcox@math.utah.edu