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Spectral and scattering theory for wave propagation in perturbed stratified media, by Ricardo Weder. Appl. Math. Sci., vol. 87, Springer-Verlag, Berlin and New York, 188 + pp., $35.00. ISBN 0-387-97357-5

Introduction

This monograph is volume 87 in the Springer series entitled *Applied Mathematical Sciences*. The series is an eclectic one with volumes on many topics in pure and applied mathematics. This volume is a contribution to modern mathematical physics as defined by the four volume work of M. Reed and B. Simon [5–8]. The monograph presents applications of abstract functional analysis to two scattering problems of classical physics. The first treats the scattering of acoustic (scalar) waves by localized inhomogeneities and/or obstacles in a plane stratified fluid. The second treats the analogous problem for electromagnetic (vector) waves.

The method used in the monograph has a history going back to work of A. Y. Povzner [4] in 1955 and T. Ikebe [1] in 1960 on quantum mechanical potential scattering. Subsequently many scattering problems of classical and quantum physics were studied by this method. They include scattering of acoustic and electromagnetic waves by localized inhomogenities and/or obstacles in homogeneous media and in cylindrical waveguides, scattering by diffraction gratings and other periodic structures, and many others. A review of the literature up to the late 1970s may be found in Reed and Simon III [7]. Recently this kind of work has gone out of fashion, although many difficult and interesting problems remain unsolved. An exception is the study of inverse scattering problems, which has been a very active field in the last decade.

The results developed in the monograph can be described in the formalism of the abstract theory of scattering with two Hilbert spaces [3]. There one has a pair of Hilbert spaces, $\mathscr{H}_0$ and $\mathscr{H}$, together with corresponding selfadjoint operators, A_0 and A, and a bounded linear bijection $J: \mathscr{H}_0 \to \mathscr{H}$. The principal objects of study are the wave operators W_+ and W_- and the scattering operator $S = W_+^* W_-$. The wave operators are formally defined by

$$W_\pm = s - \lim_{t\to\pm\infty} e^{itA} J e^{-itA_0} P_0 , \tag{1}$$

where P_0 is the orthogonal projection of $\mathscr{H}_0$ onto the subspace of absolute continuity of A_0 and $s - \lim$ denotes the strong limit. It is known that, when they exist, W_+ and W_- are partial isometries [2] with initial set $\mathscr{H}_{0,ac} = P_0\mathscr{H}_0$ and final set contained in the analogous subspace $\mathscr{H}_{ac}$ for A. W_+ and W_- are said to be complete if their final sets are precisely $\mathscr{H}_{ac}$. If W_+ and W_- are complete then S is unitary in $\mathscr{H}_{0,ac}$ [2].

Description of the work

Following a brief (two page) introduction, the monograph consists of two major chapters (Chapters 2 and 3) plus two technical appendices and a two page discussion of related literature. Chapter 2 develops spectral and scattering theory for acoustic waves in inhomogeneous fluids that are local perturbations of plane stratified fluids. Chapter 3 develops a parallel theory for electromagnetic waves. Here, for brevity, only the acoustic case will be discussed.

The monograph treats acoustic wave propagation in an inhomogeneous fluid that is governed by the wave equation

$$u_{tt}(x, y, t) - c^2(x, y)\Delta u(x, y, t) = 0, \tag{2}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, $t \in \mathbf{R}$, and Δ is the $(n+1)$-dimensional Laplacian in the coordinates (x, y). The function $c(x, y)$ is interpreted as the sound speed at position (x, y). Equations of the form (2) are studied that are perturbations of plane stratified fluids, governed by the equation

$$u_{tt}(x, y, t) - c_0^2(y)\Delta u(x, y, t) = 0, \tag{3}$$

with sound speed $c_0(y)$ at depth y. It is assumed that $c(x, y)$ and $c_0(y)$ have bounds $0 < c_m \le c(x, y)$, $c_0(y) \le c_M < \infty$. Moreover, to lighten the technical difficulties, the proofs are limited to the case of the Pekeris profile

$$c_0(y) = \begin{cases} c_+ & \text{for } y \ge h, \\ c_h & \text{for } 0 \le y < h, \\ c_- & \text{for } y < 0, \end{cases}$$

where $c_h < c_+ \le c_-$. This restriction is removed in the treatment of the electromagnetic case.

Spectral and scattering theory are developed for equations (2) and (3) under the assumption that $c(x, y)$ and $c_0(y)$ are close at infinity in the sense that for all $(x, y) \in \mathbf{R}^{n+1}$ one has

$$|c(x, y) - c_0(y)| \le C(1 + |x| + |y|)^{-\alpha}, \qquad \alpha > 1.$$

The formalism of the abstract theory of scattering is introduced by defining the operator A_0 to be a selfadjoint realization of the operator

$$A_0 = -c_0^2(y)\Delta$$

in the Hilbert space $\mathscr{H}_0$ with scalar product

$$(u, v)_{\mathscr{H}_0} = \int_{\mathbf{R}^{n+1}} u(x, y)\overline{v(x, y)}c^{-2}(x, y)_0(y)\, dx\, dy,$$

while the operator A is a selfadjoint realization of the operator

$$A = -c^2(x, y)\Delta$$

in the Hilbert space $\mathscr{H}$ with scalar product

$$(u, v)_{\mathscr{H}} = \int_{\mathbf{R}^{n+1}} u(x, y)\overline{v(x, y)}c^{-2}(x, y)\, dx\, dy.$$

The pair $(\mathscr{H}, A)$ is appropriate for the Cauchy problem for the wave equation (2). The monograph also treats the initial-boundary value problem for (2) with (x, y) in an exterior domain Ω ($\mathbf{R}^{n+1} - \Omega$ bounded). In this case functions in the domain of A are required to satisfy a Dirichlet, Neumann, or Robin boundary condition.

The starting point for the research presented in this monograph is the reviewer's monograph [9]. There a complete spectral analysis and generalized eigenfunction expansion were developed for the operator A_0. The spectral theorem and generalized eigenfunctions were used to construct a unitary operator

$$F: \mathscr{H}_0 \to \widehat{\mathscr{H}} = \bigoplus_{j=0}^{\infty} \widehat{\mathscr{H}}_j,$$

which was spectral for A_0 in the sense that $FA_0\varphi = \lambda F\varphi$ for all φ in the domain of A_0. F is called the generalized Fourier map for A_0.

The primary goal of the monograph is to prove the existence and completeness of the wave operators $W_\pm$ for the pair A_0, A. The main steps of the proof may be described briefly as follows.

Step 1. The spectral theorem and the generalized Fourier map F are used to prove that the resolvent operator

$$R_0(z) = (A_0 - z)^{-1}$$

has boundary values $R_0(\lambda \pm i0)$ at the points

$$\lambda \in [0, \infty) = \text{ spectrum of } A_0.$$

In scattering theory this is called the limiting absorption principle. The convergence of $R_0(z)$ is in the uniform operator topology of bounded operators between certain weighted Sobolev spaces (and not in $\mathscr{H}_0$).

Step 2. The limiting absorption principle for A_0 and a local compactness theorem for the resolvent of A,

$$R(z) = (A - z)^{-1},$$

are used to prove a limiting absorption principle for A; that is, the existence of the limits $R(\lambda \pm i0)$.

Step 3. The operators $R(\lambda \pm i0)$ and the spectral theorem are used to define generalized Fourier maps F_+ and F_- for A such that

$$F_\pm : \mathscr{H}_{ac} \to \widehat{\mathscr{H}},$$

and F_+ and F_- are unitary and spectral for A: $F_\pm A\varphi = \lambda F_\pm \varphi$.

Step 4. Using the preceding results, it is shown that W_+ and W_- are given by the generalized Fourier maps as

$$W_\pm = F_\pm^* F. \tag{4}$$

This representation and the properties of F and $F_\pm$ described above imply that W_+ and W_- are complete.

The S Matrix. Equations (4) imply that the scattering operator $S = W_+^* W_-$ has the representation

$$S = F^* \widehat{S} F, \quad \text{where } \widehat{S} \text{ is defined by } \widehat{S} = F_+ F_-^*.$$

$\widehat{S}$ is a unitary operator in $\widehat{\mathscr{H}}$. In the final portion of Chapter 2 it is shown that $\widehat{\mathscr{H}}$ has a direct integral structure

$$\widehat{\mathscr{H}} = \bigoplus \int_0^\infty \widehat{\mathscr{H}}(\lambda)\, d\lambda,$$

which reduces $\widehat{S}$ so that

$$\widehat{S} = \bigoplus \int_0^\infty S(\lambda)\, d\lambda .$$

The existence of the operators $S(\lambda)$ is proved, via the limiting absorption principle for A, and it is shown that $S(\lambda) - I$ is compact and the mapping $\lambda \to S(\lambda)$ is locally Hölder continuous with Hölder exponent $\gamma < 1/2$. The chapter ends with some results on the continuous dependence of the wave and scattering operators on the perturbation.

Intended readers. The brief and highly simplified description above omits mention of many difficulties the author overcame in preparing this work. In fact, the work is intricate and fraught with technical difficulties throughout. Moreover, the monograph is written in a terse, fast-paced style and employs much of the functional analytic machinery of modern mathematical physics. Thus, in order to follow the exposition in detail one needs to have a working knowledge of a substantial portion of the Reed-Simon books [5–8] or equivalent material. Mathematical physicists and mathematicians with this background should find the work stimulating and useful.

The introduction to the monograph states that the work is also intended for applied mathematicians, physicists, and engineers. The usefulness of the work to these groups is doubtful. Applied mathematicians have traditionally disliked and shunned what they regard as the overly cumbersome methods of functional analysis. Also, the subject is not a part of most physics and engineering curricula. These readers will find the monograph very difficult to penetrate.

Concluding remarks. The monograph is largely based on the author's own research during the late 1980s. Many of the results in it are new or are proved for the first time: see the Notes. These include the results on the existence and Hölder continuity of the trace maps associated with the unperturbed acoustic propagator (Lemmas 2.1 and 2.2), and the results on the absence of positive eigenvalues for the perturbed acoustic propagator (Theorems 5.1 and 5.4). These results, and the techniques for proving them, should have applications to many other scattering problems, old and new.

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Almost free modules. Set-theoretic methods, by Paul C. Eklof and Alan H. Mekler. North-Holland, Amsterdam, New York, Oxford, and Tokyo, 1990, 481 pp., $115.50. ISBN 0-444-88502-1

Forty years ago, J. H. C. Whitehead asked whether or not an abelian group A had to be free (i.e., free abelian) if all abelian extensions of the group $\mathbb{Z}$ of integers by A were splitting [in other words, if $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$]. This purely group theoretical question was motivated by problems outside the realm of group theory (the second Cousin problem, see Stein [7], and questions raised by Dixmier [2]). For countable abelian groups A, Stein [7] and Ehrenfeucht [4] gave affirmative answers, but the solution for groups of higher cardinalities looked extremely difficult, as witnessed by several publications, which contained only fragmentary results in the general case. The Whitehead problem remained for a while one of the handful of major open problems in the theory of abelian groups.

In 1973, a young mathematician, Saharon Shelah, got interested in the problem. He had the bright idea of approaching the problem from a different angle by scrutinizing the underlying sets. He was able to prove that already for groups A of cardinality $\aleph_1$, the Whitehead problem was undecidable in ZFC (the Zermelo-Fraenkel axioms of set theory plus the Axiom of Choice). More precisely, in the constructible universe L, $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ implies that A has to be free, while in models in which the Continuum Hypothesis fails but Martin's Axiom holds, there do exist nonfree groups A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$. This unexpected result was a big surprise and drew immediately the attention to the relevance of set-theoretical techniques in solving purely algebraic problems.

Shelah's discovery marked the beginning of the modern era of applications of powerful set-theoretical methods in algebra. Since then a great deal of significant work has been done in the area. The systematic use of additional set-theoretical hypotheses led to new insight into (and sometimes to a solution of) several open problems in algebra—as it did in other fields of mathematics. Shelah has remained the leading force in the developments, providing leadership and continuous stimulus to the subject.

The book by Eklof and Mekler under review presents an excellent up-to-date and in-depth survey of most of the recent developments in the area. The authors—who themselves have been at the forefront of the developments—have written a book, which is a fine example of how two different fields of mathematics can interact and create a new, flourishing field. (Actually, the title