9. C. H. Wilcox, *Sound propagation in stratified fluids*, Appl. Math. Sci., vol. 50, Springer-Verlag, Berlin and New York, 1984.

CALVIN H. WILCOX
UNIVERSITY OF UTAH
E-mail address: wilcox@math.utah.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

Almost free modules. Set-theoretic methods, by Paul C. Eklof and Alan H. Mekler. North-Holland, Amsterdam, New York, Oxford, and Tokyo, 1990, 481 pp., $115.50. ISBN 0-444-88502-1

Forty years ago, J. H. C. Whitehead asked whether or not an abelian group A had to be free (i.e., free abelian) if all abelian extensions of the group $\mathbb{Z}$ of integers by A were splitting [in other words, if $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$]. This purely group theoretical question was motivated by problems outside the realm of group theory (the second Cousin problem, see Stein [7], and questions raised by Dixmier [2]). For countable abelian groups A, Stein [7] and Ehrenfeucht [4] gave affirmative answers, but the solution for groups of higher cardinalities looked extremely difficult, as witnessed by several publications, which contained only fragmentary results in the general case. The Whitehead problem remained for a while one of the handful of major open problems in the theory of abelian groups.

In 1973, a young mathematician, Saharon Shelah, got interested in the problem. He had the bright idea of approaching the problem from a different angle by scrutinizing the underlying sets. He was able to prove that already for groups A of cardinality $\aleph_1$, the Whitehead problem was undecidable in ZFC (the Zermelo-Fraenkel axioms of set theory plus the Axiom of Choice). More precisely, in the constructible universe L, $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ implies that A has to be free, while in models in which the Continuum Hypothesis fails but Martin's Axiom holds, there do exist nonfree groups A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$. This unexpected result was a big surprise and drew immediately the attention to the relevance of set-theoretical techniques in solving purely algebraic problems.

Shelah's discovery marked the beginning of the modern era of applications of powerful set-theoretical methods in algebra. Since then a great deal of significant work has been done in the area. The systematic use of additional set-theoretical hypotheses led to new insight into (and sometimes to a solution of) several open problems in algebra—as it did in other fields of mathematics. Shelah has remained the leading force in the developments, providing leadership and continuous stimulus to the subject.

The book by Eklof and Mekler under review presents an excellent up-to-date and in-depth survey of most of the recent developments in the area. The authors—who themselves have been at the forefront of the developments—have written a book, which is a fine example of how two different fields of mathematics can interact and create a new, flourishing field. (Actually, the title of the book is somewhat misleading as far as the theory of almost free modules is just one of the four major topics covered by the book; but the subtitle gives fair indication of what the reader can expect.)

One of the introductory chapters reviews most of the basic set-theoretical concepts needed in the volume, including measurable cardinals, ultraproducts, cubs (closed and unbounded subsets), stationary subsets, trees, etc. A later chapter is devoted to more advanced set theory: the discussion of Gödel's constructible universe L and prediction principles, which are indispensible in the applications. Ronald Jensen's Diamond Principle $\diamondsuit$ and other consequences of the Axiom of Constructibility ($V = L$) are extensively studied. In addition, axioms, which arise from Paul Cohen's forcing method and are inconsistent with $V = L$, are discussed. The most well-known axioms of this sort are Martin's Axiom and Shelah's Proper Forcing Axiom. Another chapter introduces the reader to Shelah's Black Box principles: these are $\diamondsuit$-like principles, which are, unlike $\diamondsuit$, provable in ZFC. They are rather sophisticated but awfully powerful.

When one writes a book on a fairly new and rapidly developing subject, there are a number of difficult choices to make. The authors point out that it is no longer possible to give a full account of all the results that rely heavily on set-theoretic methods even if one concentrates, as the authors did, on group and module theory. The focus is on four major algebraic themes, which are treated more thoroughly, but various other topics are also described.

1. Almost Free Modules. For a cardinal κ, a module M (over any ring with 1) is called κ-free if (roughly speaking) every subset of M whose cardinality is less than κ can be embedded in a free submodule of M, which has less than κ generators (for the precise definition see p. 83). The hardest question asks for a characterization of those cardinals κ for which there exist nonfree, κ-free modules with κ generators. In general, the answer depends on the set-theoretical hypotheses. However, for singular cardinals κ, the answer is negative in every model of ZFC, as shown by Shelah's powerful Singular Compactness Theorem, which is one of the highlights of the theory.

2. The Structure of Ext. In-depth investigations concerning the group of extensions started with the Whitehead problem, and, as one might guess, over the years the results have been generalized and a number of related questions have been investigated. Significant efforts have been made to find out more about the precise structure of Ext, not just to answer the question as to when it vanishes. The book offers lots of interesting results on what group structure Ext can have under various set-theoretical hypotheses as well as on the undecidability of the occurrence of certain other structures.

3. The Structure of Hom. The central theme is the structure of the dual groups $A^\bullet = \mathrm{Hom}(A, \mathbb{Z})$ for abelian groups A. It is easy to construct dual groups by iterating the formations of direct sums and direct products of copies of $\mathbb{Z}$, but it is much harder to find dual groups that are not of this form. Dual groups are related to the questions of reflexivity (i.e., $A^{\bullet\bullet} \cong A$ canonically), nonreflexivity, and strong nonreflexivity ($A^{\bullet\bullet} \not\cong A$). Most of the material on dual groups is new.

4. Endomorphism Rings. Which rings can appear as endomorphism rings of certain classes of abelian groups, and more generally, of modules? Historically, the first essential result is due to Corner [1], who proved that every countable reduced torsionfree ring is the endomorphism ring of some countable torsion-free abelian group. In 1982, Dugas and Göbel [3] showed that in L every

cotorsionfree ring is the endomorphism ring of a cotorsionfree abelian group. (Cotorsionfree means that the group contains no nonzero elements of finite order and no subgroup isomorphic to the additive group of the rationals or of the p-adic integers, for any prime p.) In 1984, Shelah [6] proved in ZFC a stronger claim: given a cotorsionfree ring R, there are arbitrarily large abelian groups whose endomorphism rings are isomorphic to R and are $\aleph_1$-free as R-modules.

Each chapter ends with a stimulating list of exercises—some with hints—which complete and extend the scope of the treatment of the subjects. Some of the exercises are routine, but others contain results whose proofs were considered too distracting to include in the main text. The exercises are followed by notes in which the authors have made a serious effort to provide historical information and references to the literature where further details and additional results may be found.

The book under review is designed to be accessible with only a general background in set theory but no previous knowledge of the subject. It provides new approaches to several topics and gives very readable and detailed proofs of numerous results; a number of published proofs are substantially shortened and improved, and many new theorems have been added. The presentation is carefully written in accordance with the authors' intent to make the book accessible to a larger audience.

The book is a most welcome addition to the literature. It provides an insight into a beautiful and somewhat mysterious subject: the interaction of set theory and pure algebra. The importance of understanding the role of additional axioms in solving concrete algebraic problems is emphasized by the authors. The book is a striking evidence that significant advances in answering some relevant questions can only be achieved by a clever combination of algebraic and set-theoretical approaches. The new developments have been embraced by an increasing number of algebraists, but there are many others for whom a departure from ZFC seems unacceptable. The book is a big step forward in eliminating this gap.

This thought-provoking volume covers a large range of ideas and techniques, and consequently, it has a lot to offer both to algebraists and to set theorists; but many other mathematicians may also take advantage of the ideas expressed. The book is not only a very fine introduction to a fast growing area, but it also brings the reader up to the cutting edge of research. (At the end of the book thirty research problems are listed.) Undoubtedly, it will be a valuable resource for those who are interested in the subject and will serve as the standard reference guide in the area for years to come.

References

1. A. L. S. Corner, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **13** (1963), 687–710.
2. J. Dixmier, *Quelques propriétés des groupes abéliens localement compacts*, Bull. Sci. Math. **81** (1957), 38–48.
3. M. Dugas and R. Göbel, *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **45** (1982), 319–336.
4. A. Ehrenfeucht, *On a problem of J. H. C. Whitehead concerning abelian groups*, Bull. Acad. Polon. Sci. Cl. III **3** (1955), 127–128.
5. S. Shelah, *Infinite Abelian groups, Whitehead problem and some constructions*, Israel J. Math. **18** (1974), 243–256.

6. S. Shelah, *A combinatorial theorem and endomorphism rings of abelian groups* II, Abelian Groups and Modules, CISM Courses and Lectures, no. 287, Springer-Verlag, 1987, pp. 37–86.
7. K. Stein, *Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Math. Ann. **123** (1951), 201–222.

Laszlo Fuchs
Tulane University
E-mail address: mt0mamf@vm.tcs.tulane.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

André Weil: The apprenticeship of a mathematician, by André Weil (translated from French by Jennifer Gage). Birkhäuser Verlag, Basel, Berlin, 1992, 197 pp., $29.50. ISBN 3-7643-2650-6

Early in 1947 there were two remarkable mathematical events.

On the one hand, a Colloquium volume on The Foundations of Algebraic Geometry by André Weil appeared; it presented a firm basis for a subject that beforehand seemed without real foundations, and it provided Weil's proof of the Riemann Hypothesis on the Zeta function for function fields.

On the other hand, new volumes of Bourbaki's Elements were published. It was suddenly clear that this redoubtable multicephalic author was indeed going to cover all the basic parts of mathematics and that this would vitally influence the way a whole generation would view the subject.

We all heard the legend: Cartan, Chevalley, Delsarte, Dieudonné, and Weil (The Founding members) visited Montmartre to find a bearded clochard muttering in his absinthe insights about compact structures and their representations. They then sat at his feet, learned all about it, and polished it up in elegant form. My files once had a splendid photo of that clochard, Nicholas Bourbaki, white beard and all.

Now, as Weil writes in this book, "The time has come to unveil these mysteries." Here, gentle reader, you will discover the real way in which this unusual and influential collaboration came about. You will also find a sensitive and insightful presentation of the development of this remarkable mathematician; without going into technical detail, this slim, well-written, and captivating volume shows the results of early exposure to the highly charged scientific milieu of Paris.

André Weil was born in Paris on May 6, 1906. By age 5 he had learned to read. His father was a physician; his mother closely supervised his early education, finding him special tutors, getting him to skip some forms (grades), and finding him a number of truly accomplished teachers. Weil fondly recalls several of these teachers, especially one M. Collin, who taught him in the first (top) form at the famous Lycée Saint Louis—and in particular, brought him to understand that, in writing mathematics, one should never say "it is obvious that". Weil studied Latin, Greek, and Sanskrit. With friendly advice from Hadamard, he studied Jordan's "Cours d'Analyse".