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Variational methods (Applications to nonlinear PDE and Hamiltonian systems), by Michael Struwe, Springer-Verlag, New York, 1990, 244 pp., $39.50. ISBN 3-540-52022-8.

The calculus of variations has a long history. It essentially started soon after the introduction of calculus by Newton and Leibniz, although some individual optimization, minimization, or maximization problems had been investigated before that, most notably the determination of the paths of light by Fermat. The first phase of the development of the calculus of variations was characterized by a combination of mathematical methods, philosophical concepts, and physical problems; this was started by Leibniz and pursued by Euler, Maupertuis, and others. For this reason, it also attracted much more general interest than the subject does today. While mathematical priority disputes today can be as vicious as ever but are confined to a small circle of insiders, in former times they sometimes gave rise to much public debate. One of them, about the concepts underlying the calculus of variations, even led to the philosopher Voltaire (the guy who invented the story about Newton and the apple) being thrown into jail by the Prussian King Frederick the Great (the one who started a war against every other continental European power and still came out victorious; as a result of this war, the French had to cede Canada to the British).

In mathematical terms, the first variational problems were considered by the Bernoulli brothers, and the first general theories were developed by Euler and Lagrange. In the last century, variational arguments played a prominent role, for example, in Riemann's work, but solid foundations could only be laid in this century through the work of Hilbert and many others.

For smooth functions on finite-dimensional spaces, it is often quite easy to find critical points, provided some compactness condition holds, and this is, for example, the subject of Morse theory. More difficult cases, where the critical points, for example, may be degenerate, form the subject of Conley theory.

Through arguments of Weierstraß, it was realized that the situation becomes different if one wants to find critical points for functionals on an infinite-dimensional space because the space is not locally compact anymore. To overcome the difficulty, this functional then needs to satisfy more stringent requirements.

The subject of the calculus of variations then has become to isolate such conditions and prove their sufficiency, and the purpose of the book is to discuss some such strategies. In the first of three chapters, the direct methods of the calculus of variations are treated. Here, one wants to find the minimum of a variational problem as the limit of a minimizing sequence. One needs a suitable concept of compactness that, together with the properties of the functional, ensures the existence of this limit and a lower semicontinuity theorem that guarantees that this limit indeed minimizes the functional. The author discusses several general methods to obtain such compactness (improving a minimizing sequence through additional normalizations, compensating a possible loss of compactness by suitable nonlinear constraints, certain transformations based on duality principles, and others).

In the second chapter, one tries to find unstable critical points, i.e., those of minimax type. A general condition for those to exist has been found, the so-called Palais-Smale condition, and this condition is discussed in detail. The Palais-Smale condition ensures that a minimaxing sequence indeed converges to a critical point with functional value as the limit of those of the sequence. The difficulty in verifying the Palais-Smale condition often is that the functional is only lower semicontinuous but not continuous under the type of convergence that some weak compactness gives. Also, one needs topological conditions in order to have minimaxing sequences and several such conditions are presented in the book.

The third chapter deals with limit cases of the Palais-Smale condition where it fails globally but remains valid locally. Some of the most important variational problems belong here, such as Yang-Mills fields in four dimensions, the Yamabe problem, two-dimensional harmonic maps, and minimal and more generally prescribed mean curvature surfaces, and in particular here the author has made important contributions in the past. The book discusses several of these problems, and it also makes an attempt to find some structural similarities of such problems. The author usually motivates and substantiates the general theory and the methods developed by applications to specific examples, mostly to various types of semilinear elliptic equations.

In order to get a clearer picture of the aim and the scope of the book, let us first tell what the book is not meant to be. It is not a textbook on the calculus of variations, because the material is not developed systematically, but the reader is often referred to other publications for details. It is also not a research monograph, because most of the results have been published elsewhere already (nevertheless, there are a couple of improvements over the existing literature). It is also not a complete survey of the modern calculus of variations, because the author often proceeds in a rather selective manner. Instead, the book resembles more the contents of a topics course in the calculus of variations. The author develops certain selected important methods and results in the calculus of variations and applies them to a wide range of examples.

If one selects topics the choice usually will be subjective, and so there are some instances where the reviewer could criticize the choice of topics. This will also give us the opportunity to learn somewhat more about the details of the book. One of the oldest and also most interesting variational problems is the one of finding geodesics on Riemannian manifolds. This variational problem does satisfy the Palais-Smale condition, and so there are no further analytic difficulties for finding closed geodesics. The difficulties here are more of a topological nature, namely, to find topological classes leading to closed geodesics and to exclude that higher-order saddle point constructions just lead to multiple coverings of previously found solutions. In the book the Palais-Smale condition is verified only in a special case, but not in general, although this would not have been outside the scope of the book. Also, it might have been appropriate to mention the theorem of Fet that every compact Riemannian manifold contains at least one closed geodesic. There also is the beautiful theorem of Lusternik and Schnirelman that every Riemannian metric on S^2 supports at least three non-self-intersecting closed geodesics. The author speaks much about this celebrated theorem but never describes the essential idea of the proof. The omission of the general multiplicity problem mentioned above, however, is justified, because research here has moved too far away from the methods presented in the book. An interested reader is instead referred, for example, to Klingenberg's book [K] or Bangert's survey article [B] for some further developments. Geodesics give rise to a one-dimensional variational problem and, hence, solve a system of ODE. There are natural generalizations to higher-dimensional variational problems, namely, harmonic maps and minimal surfaces. Two-dimensional harmonic maps form a borderline case of the Palais-Smale condition. The easiest example is conformal self-mapping of S^2, and it is well known that the group of conformal self-mappings of S^2 is not compact. Therefore, even a sequence of critical points of the energy functional (whose critical points are harmonic maps, and conformal maps of Riemann surfaces are particular harmonic maps) may fail to have a convergent subsequence. Therefore, one cannot expect a general critical sequence to converge to a critical point, and the Palais-Smale condition fails. This lack of compactness, however, can be completely analyzed as the splitting off of minimal 2-spheres. This was achieved by Sacks and Uhlenbeck [SaU] and in a different context independently by Wente [W]. In particular, in situations where no minimal 2-spheres exists, one gets a good existence theory for harmonic maps. Also, if $\pi_2(N) = 0$, then splitting off a minimal 2-sphere does not change the homotopy class of a harmonic map into N, and therefore, one may obtain a harmonic map as a minimum of the energy in a given homotopy class. This is a theorem of Lemaire [L] and Sacks and Uhlenbeck [SaU]. Other proofs have been found by Schoen and Yau [SY], by the author, and by the reviewer (see, e.g., [J, Chapter 4]). The author presents his own proof, which consists in studying the associated evolution or heat flow problem, and this proof happens to be the one that is least variational and most difficult among all those quoted above. It should be mentioned, however, that it can give stronger results than just the existence of energy minimizing maps (but so do the original method of [SaU] and the general argument of [J]) and that it also exhibits some interesting mathematical features. In particular, recent work of Chang, Ding, and Ye [CDY] shows that the heat flow can develop singularities in finite time (the solution can be continued past such singularities, however). Nevertheless, the reviewer feels that choosing one of the other methods would have fit more naturally into the context developed elsewhere in the book.

Related problems concern minimal surfaces or, more generally, surfaces of constant or, even more generally, of prescribed mean curvature. In the book, the Dirichlet problem for surfaces of constant mean curvature is discussed in detail as another instance of a borderline case of the Palais-Smale condition. For the Plateau problem for surfaces of constant mean curvature, in particular for the author's own important contributions to the Rellich conjecture, the reader is referred to another monograph of the author [St]. In fact, [St] complements the present one quite nicely; it is devoted to a critical point theory for minimal surfaces and ones of constant mean curvature, and it develops that theory in a methodologically coherent and systematic manner, quite in contrast to the present one, which rather impresses by the diversity of its techniques.

If harmonic maps are defined on domains of definition higher than two, then one is well beyond the realm of validity of the Palais-Smale condition, and so far, no satisfactory critical point theory for harmonic maps could be developed.

Somehow, however, this difference between the two- and the higher-dimensional situation is not made clear in the book. Incidentally, not only the work of Eells and Sampson [ES] but also that of Al′ber [A1, A2] should have been quoted for the theory of harmonic maps into manifolds of nonpositive curvature. In contrast to the general situation, at least the theory of energy minimizing harmonic maps has been developed in arbitrary dimensions, by Schoen and Uhlenbeck [SU] (and also by Giaquinta and Giusti [GG] in a somewhat different context).

The borderline case of the Palais-Smale condition, which is most spectacular in view of its applications by Donaldson, Taubes, Floer, and others, are the Yang-Mills equations in four dimensions. They are not treated in this book. For the Yamabe problem, another interesting instance, only a short outline of its final solution by Schoen [S] is given. In any case, the level of detail given varies considerably through the book.

Most applications in the book, as noted, concern various types of semilinear elliptic equations. Pohožaev's nonexistence result indeed was the first instance where a borderline case of the Palais-Smale condition led to an interesting phenomenon, and this and subsequent developments are well represented. Other results include the existence of multiple or sometimes even infinitely many solutions for such equations, for example, with suitable symmetries, and it is quite interesting to see that many diverse techniques developed in the book all have applications here, quite often due to the author himself. There are also some applications to Hamiltonian systems, although maybe fewer than the subtitle might suggest. Other important applications of the direct methods can be found in elasticity theory. In the book, the fundamental technical result of J. Ball on the weak lower semicontinuity of polyconvex functionals is derived. The resulting existence results then are left to the reader. Since Ball's results have by now been represented in detail in several other monographs [C, D, Z], I agree with the author's choice to explain the crucial point of the theory and omit the rest.

In most details, the book is carefully written, and the reviewer did not find much to criticize here. (Some minor points: There is a wrong section heading on p. 7, and also Zeidler [2] is much more than a translation of [1].)

In spite of the fact that the reviewer does not always agree with the choice of topics, the book should be very useful for those who have some basic knowledge of the calculus of variations and its applications. The author presents a wide range of methods and techniques from the forefront of research and applies them to an impressive range of often difficult examples. There is no other book that could compete here with the present one. In most cases, the author succeeds in isolating the crucial and fundamental aspects and ideas from those technicalities that usually hide them in the literature. Even experts can learn a lot from the book. As already indicated above, the book would form a good source for a topics course or a seminar on the calculus of variations. In conclusion, the reviewer can recommend the book highly.

The price is moderate, maybe not so much in absolute terms, but at least in relation to the presently prevailing standard.

REFERENCES

[A1] S. I. Al'ber, *On n-dimensional problems in the calculus of variations in the large*, Soviet Math. Dokl. **5** (1964), 700–704.

[A2] ———, *Spaces of mappings into a manifold with negative curvature*, Soviet Math. Dokl. **9** (1967), 6–9.

[B] V. Bangert, *Geodätische Linien auf Riemannschen Mannigfaltigkeiten*, Jber. DMV **87** (1985), 39–66.

[C] Ph. Ciarlet, *Mathematical elasticity*, vol. I: *Three-dimensional elasticity*, North-Holland, Amsterdam, 1988.

[CDY] K. C. Chang, W. Y. Ding, and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces.*

[D] B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, New York, 1989.

[ES] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1986), 109–160.

[GG] M. Giaquinta and E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.

[J] J. Jost, *Two-dimensional geometric variational problems*, Wiley-Interscience, Chichester, 1991.

[K] W. Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, New York, 1978.

[L] L. Lemaire, *Applications harmoniques de surfaces riemanniennes*, J. Differential Geom. **13** (1978), 51–78.

[SaU] J. Sacks and K. Uhlenbeck, *On the existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981), 1–24.

[S] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), 479–495.

[SU] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), 307–335; Correction in J. Differential Geom. **18** (1983), 329.

[Sy] R. Schoen and S. T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math. (2) **110** (1979), 127–142.

[St] M. Struwe, *Plateau's problem and the calculus of variations*, Princeton Univ. Press, Princeton, NJ, 1989.

[W] H. Wente, *Large solutions to the volume constrained Plateau problem*, Arch. Rational Mech. Anal. **75** (1980), 59–77.

[Z] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. IV: *Applications to mathematical physics*, Springer-Verlag, New York, 1988.

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Differential topology and quantum field theory, by Charles Nash. Academic Press, New York, 1991, 386 pp., $59.95. ISBN 0-12-514075-4.

Over the last fifteen years there has been a renaissance in the interaction between geometry and theoretical particle physics. Here "geometry" includes but is not limited to: the differential geometry of connections, now called "gauge