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Around Burnside, by A. I. Kostrikin. Ergeb. Math. Grenzgeb. (3), vol. 20, Springer-Verlag, New York, Berlin, and Heidelberg, 1990, 220 pp., $82.00. ISBN 0-387-50602-0

The restricted Burnside problem, by Michael Vaughan-Lee. London Math. Soc. Monographs (N.S.), vol. 5, Oxford University Press, Oxford, 1990, 209 pp., $57.50, ISBN 0-19-853573-2

In a now-famous paper of 1902 Burnside asked some questions that have been very influential in the development of the theory of groups. Chandler and Magnus in their *The history of combinatorial group theory* [7, p. 47] go as far as to say: "A comparison of the influence of Burnside's problem on combinatorial group theory with the influence of Fermat's last theorem on the development of algebraic number theory suggests itself very strongly." Recall that a group has exponent e if the eth power of every element is the identity. The central question can then be stated: Given positive integers d and e, is every group that has exponent e and can be generated by d elements finite? As the result of work by Novikov and Adyan (see Adyan's monograph [1]) the answer is in general no. Recently there have been announcements (by S. V. Ivanov and by Lysënok) that show that (when $d \geq 2$) the answer is no, except for finitely many e.

The two books under review have as their common theme a related question usually referred to as the restricted Burnside problem: Is there among the finite

d-generator groups with exponent e a largest one? This question is first stated in a paper [17] by Magnus in 1950—it was current on the seminar circuit already in the 1930s. The 1950s saw major advances toward its solution. The highlights were: the reduction theorem of Hall and Higman [10] in which they reduced the question to the case of prime power exponent and to a question about finite simple groups (the Schreier conjecture); and the positive answer by Kostrikin [13] for the case of prime exponent. As Kostrikin says in the preface to his book: "The theorem is still far from being a triviality. Furthermore, there has not grown out of it a solution of RBP for arbitrary prime-power exponents p^k, as was expected." Moreover he felt "the need to make available to experts in the area a text that is easily checked and does not pretend to the deceptive brevity of the original paper. All errors to be found there have been subjected to impartial analysis, and, it is hoped, have been eliminated". In fact the book has been dramatically successful in stimulating further work that has resulted in the positive answer by Zel′manov [21, 22] for the case of prime power exponent. This combined with the affirmative resolution of the Schreier conjecture (see, for example, Aschbacher et al. [4]) completes the solution of the restricted Burnside problem.

The Russian original of Kostrikin's monograph appeared in 1986. The English translation, which incorporates several improvements and additions, gives a clear and accessible account of the ideas that went into the original 1950s proof and highlights those that can now be seen to be central to the enterprise. It also treats some of the newer developments.

The problem is reduced to a question about Lie rings by associating with every finite group of prime exponent a Lie ring. Let the Lie multiplication be denoted by $[\ ,\ ]$. A Lie ring is said to satisfy the nth Engel condition if for all x, y the left-normed Lie monomial $[x, y, \dots, y]$ with n y's is 0. Kostrikin's main result is that a finitely generated Lie ring of prime characteristic p satisfying the $(p-1)$th Engel condition is nilpotent and (hence) finite dimensional. Since the Lie rings associated with groups of exponent p have characteristic p and satisfy the $(p-1)$th Engel condition, there is an upper bound on the order of d-generator finite groups of exponent p. Zel′manov has now proved that a Lie ring of prime characteristic which satisfies the linearized form of an Engel condition and a restricted Engel condition in which the y runs only over Lie monomials is nilpotent. It follows, in much the same way, that the restricted Burnside problem for prime power exponent has a positive solution. A key notion in both proofs is what is best explained as an "element with the sandwich property" but for brevity is simply called a *sandwich*; this is an element a such that $[a, x, a] = [a, x, y, a] = 0$ for all x, y. Kostrikin [14, 15] proved that a Lie ring of prime characteristic p which satisfies the $(p-1)$th Engel condition and is generated by finitely many sandwiches is nilpotent. Zel′manov and Kostrikin [23] have sharpened this to: every Lie ring generated by a finite set of sandwiches is nilpotent. Zel′manov's decisive contribution to the restricted Burnside problem is a remarkable tour de force that combines ideas from different areas of mathematics with exceptional technical skill. It draws its inspiration both from the intricately detailed methods set out in Kostrikin's monograph and from the theory of Jordan algebras. (A later simplification of Zel′manov's proof by Vaughan-Lee has freed it from its reliance on results about Jordan algebras.) A striking corollary of Zel′manov's work deserving mention here is that every finitely generated Lie ring satisfying an Engel condition is nilpotent.

Kostrikin's original proof was a grand proof by contradiction and his monograph follows the same path. In fact as Adyan and Razborov [2] have shown (spurred on by refereeing the Russian original of the monograph) this proof can be made effective and used to give a primitive recursive upper bound for the orders of the largest d-generator groups of prime exponent. Another effective proof, by Zel′manov, is given in an appendix to the translation of Kostrikin's monograph. More recently Vaughan-Lee and Zel′manov have announced an upper bound for the order of the largest finite d-generator group of exponent p^k which can be written explicitly as an exponential tower in the terms of d and p^k. It should be said that such bounds are hardly likely to be sharp.

The restricted Burnside problem can also be stated as a finiteness problem. A group is called residually finite if every nonidentity element has a nonidentity image in some finite quotient of the group. Then the solution can be stated: a finitely generated residually finite group of finite exponent is finite. Burnside's 1902 paper [6] opened with a question that is usually interpreted as: Is every finitely generated periodic group finite? (A group is called *periodic* when every element has finite order.) Even a restricted version of this has a negative answer: There are infinite finitely generated periodic residually finite groups. This was first shown by Golod [8] and many other examples have been constructed. So the restricted Burnside problem has been shown to be the appropriate finiteness question in this context. As Burnside himself already pointed out the positive answer leads to the supplementary question: What is the order of the largest finite d-generator group of exponent e? (This group is $R(d, e)$ in Vaughan-Lee's notation and $B_0(d, e)$ in Kostrikin's notation.) A general solution appears to remain far out of reach. Burnside gave the easy answer for exponent 2, Levi and van der Waerden [16] gave the answer for exponent 3, and Hall and Higman [10] gave the answer for exponent 6. Beyond this the orders are known only for a few "small" cases: for exponent 4 up to 5 generators, the order of $R(5, 4)$ is 2^{2728}; and for exponent 5 up to 3 generators, the order of $R(3, 5)$ is 5^{2282} (Vaughan-Lee [20]).

For the case of prime power exponent the groups $R(d, p^k)$ are nilpotent and so one can ask, as an intermediate question: What is the nilpotency class of $R(d, p^k)$? Razmyslov [18, 19] has given examples to show that for $p^k \geq 4$ there is no global upper bound on the class independent of d. These examples combined with a result of Gupta and Newman [9] give that the class of $R(d, 4)$ is $3d - 2$ for $d \geq 3$. For exponent 5 Higman [12] gave a linear upper bound for the class since sharpened to $6d$ (Havas, et al. [11]) and there is a lower bound of $2d - 1$ (Bachmuth, et al. [5]) but the exact class is not known except for $d = 2, 3$ (12 and 17, respectively). An important observation, by Adyan and Repin [3], is that for sufficiently large primes (and fixed $d \geq 2$) there is an exponential lower bound for the class—a neat proof of this is given in Kostrikin's monograph.

Although the two books under review cover much common ground there are differences in outlook and emphasis. As mentioned earlier, Kostrikin's central concern is to give an accurate and fully detailed account of the ideas and techniques that went into the original proof of his theorem. But in addition to this he has consciously fashioned his theory as a working tool and presented it as a springboard for further developments. It was meant to be an influential book and so it has proved to be. As the translator comments, the style is unusual and there are numerous allusions and witticisms—there is talk of a Trojan Horse and of witch doctors' incantations. The book will continue to provide a valuable entry to its subject despite the dramatic recent advances.

Vaughan-Lee's choice of topics is rather different from Kostrikin's and in part reflects his own very substantial contributions to the subject. Thus, greater emphasis is placed on the interrelations of groups and Lie rings and their group-theoretical consequences. Again, the use of computers plays an essential part in the theoretical development. These features are skillfully combined in the author's definitive treatment of groups of exponent 4. There are some differences too in the treatment of common topics: The proof of Kostrikin's theorem while following one of the paths marked out by Kostrikin is both constructive and relatively short. Vaughan-Lee writes concisely, keeping side issues to a minimum, but never glosses over the details. In all, despite considerable common ground, the two books supplement rather than duplicate one another.

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Nonlinear partial differential equations of second order, by Guangchang Dong, Transl. Math. Monographs, Amer. Math. Soc., Providence, RI, 1991, vii + 249 pp., $136.00. ISBN 0-8218-4554-3

The study of nonlinear partial differential equations in this century has been dominated by one basic idea: a priori estimates. Bernstein was one of the early pioneers in this field, proving many estimates in the first decade of the century, and Schauder's work in the thirties still guides most modern attempts at proving existence via estimates. In particular, the Leray-Schauder fixed point theorem reduces the question of existence of solutions to the Dirichlet problem for quasilinear elliptic equations (or the initial Dirichlet problem for parabolic equations) to four estimates on solutions to a family of related problems. If u is the solution in the domain Ω, we estimate $\|u\|_{\infty,\Omega}$, $\|Du\|_{\infty,\partial\Omega}$, $\|Du\|_{\infty,\Omega}$, and $|Du|_{\alpha,\Omega}$ for some positive α, where this last norm is just the usual Hölder norm and D denotes the spatial gradient. When the boundary condition is nonlinear with respect to the gradient, a different fixed point theorem is needed even for quasilinear equations, but the same estimates arise. For fully nonlinear equations, the latter fixed point theorem is also used and the corresponding norms of second derivatives of u are estimated.

The development of appropriate a priori estimates follows a long trail, but there are important dates to be noted. After the early work of the nineteenth century, in which only simple equations such as the usual heat equation were considered, Bernstein, in the early years of the twentieth century, proved some basic estimates for elliptic equations, and many of these estimates have simple parabolic analogs. In the thirties, Schauder developed his theory of linear elliptic equations and some important ideas for studying general nonlinear equations. In the fifties, Barrar and Friedman proved parabolic analogs of Schauder's linear estimates. A crucial turning point for studying nonlinear elliptic and parabolic equations came in 1957 and 1958 with the discovery of Hölder estimates for