APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993, Pages 161-164

BOOK REVIEW

Nonlinear partial differential equations of second order, by Guangchang Dong, Transl. Math. Monographs, Amer. Math. Soc., Providence, RI, 1991, vii + 249 pp., $136.00. ISBN 0-8218-4554-3

The study of nonlinear partial differential equations in this century has been dominated by one basic idea: a priori estimates. Bernstein was one of the early pioneers in this field, proving many estimates in the first decade of the century, and Schauder's work in the thirties still guides most modern attempts at proving existence via estimates. In particular, the Leray-Schauder fixed point theorem reduces the question of existence of solutions to the Dirichlet problem for quasilinear elliptic equations (or the initial Dirichlet problem for parabolic equations) to four estimates on solutions to a family of related problems. If u is the solution in the domain Ω, we estimate $\|u\|_{\infty,\Omega}$, $\|Du\|_{\infty,\partial\Omega}$, $\|Du\|_{\infty,\Omega}$, and $|Du|_{\alpha,\Omega}$ for some positive α, where this last norm is just the usual Hölder norm and D denotes the spatial gradient. When the boundary condition is nonlinear with respect to the gradient, a different fixed point theorem is needed even for quasilinear equations, but the same estimates arise. For fully nonlinear equations, the latter fixed point theorem is also used and the corresponding norms of second derivatives of u are estimated.

The development of appropriate a priori estimates follows a long trail, but there are important dates to be noted. After the early work of the nineteenth century, in which only simple equations such as the usual heat equation were considered, Bernstein, in the early years of the twentieth century, proved some basic estimates for elliptic equations, and many of these estimates have simple parabolic analogs. In the thirties, Schauder developed his theory of linear elliptic equations and some important ideas for studying general nonlinear equations. In the fifties, Barrar and Friedman proved parabolic analogs of Schauder's linear estimates. A crucial turning point for studying nonlinear elliptic and parabolic equations came in 1957 and 1958 with the discovery of Hölder estimates for solutions of linear divergence structure equations with bounded measurable coefficients by deGiorgi and Nash. These estimates were extended to quasilinear divergence structure equations by Ladyzhenskaya and Ural'tseva in the early 1960s. In the late sixties, Serrin proved boundary gradient estimates for a large class of nonuniformly elliptic equations and the parabolic versions of these estimates were proved in the early 1970s by Edmunds and Peletier, Ivanov, and Trudinger. The next crucial step occured in 1979 when Krylov and Safonov proved a Hölder estimate for solutions of linear nondivergence equations with bounded measurable coefficients. This discovery opened the possibility of analyzing so-called fully nonlinear equations. Since 1980, the field has

0273-0979/93 $1.00 + $.25 per page

virtually exploded. Estimates once thought unobtainable are being proved every day, and it seems that the only real limitation to current research on a priori estimates is what estimates are true. In the course of this review, we shall mention some of the more recent results in some detail.

The book being reviewed here presents results concerning these a priori estimates and their applications. Although most of the book studies parabolic equations, there are also chapters on hyperbolic equations and on the Schrödinger equation. A major strength of this book is that it introduces the West to many results from China; a quick perusal of the references yields many papers only available in Chinese. In addition, there are many Western and Russian works quoted.

Care must be taken in reviewing any book that attempts to portray the cutting edge of research. This book in particular suffers from the obvious pitfalls inherent in such an endeavor. Many of the estimates in this book have been improved between the original writing and the English translation, although one should not fault the author too heavily for such omissions. After all, it is difficult to keep up with all of the latest developments. Also, as in all branches of mathematics, in estimate-proving major improvements to one person are trivial consequences of known results to another. To demonstrate some of these estimates, let us look at some examples, with commentary.

First are the degenerate parabolic equations from Chapter V. Here the problem is to solve the equation

$$u_t = D_i(a^{ij}(x,t,u)D_ju) + b^i(x,t,u)D_iu + c(x,t,u) = 0$$

in some cylindrical domain $Q = \Omega \times (0,T)$, where repeated indices are summed from 1 to n, $\Omega \subset \mathbb{R}^n$, the coefficients a^{ij} and b^i are continuously differentiable with respect to x, $D = (D_1, \dots, D_n)$ denotes the spatial gradient operator, and a^{ij} satisfies the inequalities

$$\tfrac{1}{\Lambda}\nu(|r|)|\xi|^2 \le a^{ij}(x,t,u)\xi^i\xi^j \le \Lambda\nu(|r|)|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and $(x,t,r) \in Q \times \mathbb{R}$, where Λ is a positive constant and the function ν satisfies the inequality

$$1 \le \frac{r\nu(r)}{\int_0^r \nu(s)\,ds} \le m$$

for all positive, sufficiently small r. This inequality on ν includes as a special case $\nu(r) = r^{m-1}$ for $m \ge 1$ a constant. Dong proves, under some additional technical hypotheses on the equation, that solutions exist and are Hölder continuous. In fact, Sacks [9] and DiBenedetto [1] had shown in the early 1980s that solutions exist and are continuous under much weaker conditions on ν and without any differentiability of a^{ij} or b^i. Actually, DiBenedetto and Sacks studied slightly different classes of equations, but their ideas work for this structure as well. The step from continuity to Hölder continuity is nontrivial, but it is a simple corollary of later work of DiBenedetto and Friedman [3], who explicitly proved the Hölder continuity in the case where $\nu(r) = r^{m-1}$ with $m > 0$.

In §3 of Chapter I, Dong follows the much earlier book of Ladyzhenskaya, Solonnikov, and Ural'tseva [6] (the first and third names are transliterated as Ladyženskaja and Ural′ceva in this reference) in studying for solutions of divergence structure equations:

$$u_t = \text{ div } A(x,t,u,Du) + B(x,t,u,Du).$$

The basic hypothesis is that $a^{ij}(x,t,z,p) = \partial A^i(x,t,z,p)/\partial p_j$ satisfies the inequality

$$\lambda(x,t,z,p)|\xi|^2 \le a^{ij}(x,t,z,p)\xi_i\xi_j \le \Lambda(x,t,z,p)|\xi|^2$$

for positive functions Λ and λ. Dong gives a synopsis of the work of Ladyzhenskaya and Ural'tseva in the case where Λ and λ are constants. The main goal (as already mentioned) is to show a Hölder gradient bound for solutions. For more general Λ and λ, such estimates are still known although none are mentioned in this book. In particular, the "unsolved problem" (Comment 6 on page 29) in which the constants are replaced by constant multiples of $1 + |p|^{m-2}$ (with $m \ge 2$ a constant) was already completely solved by 1986. The appropriate interior and global gradient estimates were proved by the reviewer [7] (under more general hypotheses) in 1983, assuming a modulus of continuity; the modulus of continuity was proved in 1986 by DiBenedetto [2]. Alternative hypotheses under which these estimates are valid can be found in [4] or [8].

The organization of the book is somewhat puzzling to the reviewer. It is a hybrid of sorts between a report of the author's work and a unified description of the field. In some ways, *Nonlinear partial differential equations of second order* is like a book of conference proceedings. There is common ground uniting all chapters, but I never got a sense of progession from one chapter to the next. Instead, Chapters I–IV are separate stories, Chapters V and VI form another little story, and then Chapters VII–X tell a final story, disjoint from the previous ones. Nowhere is this separation clearer than in the Hölder gradient estimate for solutions of quasilinear equations near the lateral boundary. A crucial element of the proof by Ladyzhenskaya and Ural'tseva is an L^∞ estimate for u_t, which assumes the coefficients A and B to be Lipschitz with respect to t. This Lipschitz dependence is not needed as demonstrated by Krylov's boundary estimate, which is Theorem 18 of Chapter IX in Dong's book. Even more puzzling is the absence of Krylov's book [5], which appeared in 1985 (with English translation in 1987), from the references.

It would be unfair to close this review with a complaint about this book, so now I will tell what I liked. First and foremost, it collects the author's large body of work on the subject into a single source. Second, the book is very well written (which is also a compliment to the translator, Kaising Tso); in each chapter, there is a clear sense of direction and the English is excellent, far better than what is found in most current mathematical journal articles. There are two audiences for whom this book is very useful: Those who want an overview of the current (as of 1985) state of affairs in parabolic equations and those interested in seeing what has been done in the field by Chinese mathematicians.

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