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Non-abelian harmonic analysis. Applications of $SL(2, \mathbb{R})$, by Roger Howe and Eng-Chye Tan. Universitext, Springer-Verlag, New York, 1992, xv+257 pp. ISBN 0-387-97768-6

Much of the power of linear algebra resides in the idea that any linear transformation on any vector space should be built from one basic example: multiplication by a scalar on a one-dimensional space. In the context of analysis, one of the most fundamental instances of this idea is the linear transformation of differentiation on smooth functions on the real line. Differentiation acts by a scalar (multiplication by λ) on each of the exponential functions $x \mapsto e^{\lambda x}$. The Fourier and Laplace transforms are used to express more general functions in terms of exponentials and so to study their derivatives.

When several linear transformations appear at the same time, the simplest possibility is that they all commute with each other (as happens, for example, with constant-coefficient differential operators on $\mathbb{R}^n$). The corresponding basic example has several operators acting by multiplication by different scalars in a one-dimensional space. Reducing problems to this case amounts to studying joint eigenvectors for the linear transformations.

Of course it often happens that one is interested in several noncommuting linear transformations at the same time. A typical example is the three operators D, X, and I acting on smooth functions on $\mathbb{R}$: differentiation, multiplication by x, and the identity. There are no eigenfunctions for X (the eigendistributions are Dirac delta functions) and so, in particular, no joint eigenfunctions for the three operators. Something more sophisticated is needed as a basic example.

To see what that might be, notice first that the linear span of D, X, and I is closed under formation of commutators. Writing $[T, S] = TS - ST$ whenever S and T are linear transformations of the same vector space, we have

$$(1) \qquad [D, X] = I, \qquad [D, I] = 0, \qquad [X, I] = 0.$$

These relations can be formulated for any three linear transformations $\{A, B, C\}$ of a vector space V:

$$(2) \qquad [A, B] = C, \qquad [A, C] = 0, \qquad [B, C] = 0.$$

Now the idea is to study linear transformations satisfying (2), in order to find what the simplest possibilities are and, having understood those simplest possibilities, to try to understand our operators on smooth functions in terms of them.

It turns out that three linear transformations of a finite-dimensional vector space cannot satisfy (2) unless C is zero (a case that is unrelated to our original example where C is the identity). To study (2) on infinite-dimensional spaces, one approach is to bring some analysis into the picture. The Stone-von Neumann theorem says that if A and B are (possibly unbounded) selfadjoint operators on a Hilbert space H satisfying (2) with C equal to the identity, then H is isomorphic to a sum of copies of $L^2(\mathbb{R})$. The isomorphism may be chosen in such a way that A and B correspond to D and X. This is a beautiful and important fact, but in the present context it suggests only that our example cannot be further simplified.

An algebraic approach is a little more flexible. The simplest model for (2) (still with $C \neq 0$) has V equal to the space of all polynomials in one variable x, A equal to differentiation, B multiplication by x, and C equal to the identity. A related possibility is to fix complex numbers r and s and to consider the space V of all finite formal sums

$$\sum_{m \equiv r \pmod{\mathbb{Z}}} a_m (x-s)^m ,$$

on which A and B act as differentiation and multiplication by x (and C acts by the identity). Yet another is to consider all polynomial multiples of a fixed exponential function. Because A and B appear almost symmetrically in (2), we can build more examples from these simply by replacing B by A and A by $-B$. Each of the resulting models of (2) can be interpreted in terms of (polynomial coefficient) differential equations on the line. This approach to differential equations (by a study of abstract algebraic relations like (2)) is at the heart of the algebraic theory of $\mathscr{D}$-modules, as developed by J. Bernstein and others (see [2, 1]).

Here is a general context in which similar ideas apply. Suppose we have a complex vector space $\mathscr{F}$ (e.g., a space of functions) and a finite set $\{D_1, \ldots, D_N\}$ of linear transformations of $\mathscr{F}$ (e.g., differential operators). Assume that the commutator of any two of these operators is a linear combination of them, namely,

$$[D_i, D_j] = \sum_k c_{ij}^k D_k \tag{3}$$

for some complex numbers c_{ij}^k. This says that the span of the D_i is a Lie algebra $\mathfrak{g}$, with Lie bracket given by the commutator of operators. In order to study the action of these linear transformations on $\mathscr{F}$, one can study first the structure of an abstract vector space V endowed with linear transformations $\{A_1, \ldots, A_N\}$, subject to the relations

$$[A_i, A_j] = \sum_k c_{ij}^k A_k . \tag{4}$$

If the D_i are linearly independent, the pair $(V, \{A_i\})$ is exactly a representation of the Lie algebra $\mathfrak{g}$, and this scheme is more or less what is meant by "abstract harmonic analysis" for Lie algebras.

To be concrete again, let us take for $\mathscr{F}$ the space of smooth functions (or polynomials, or distributions) on $\mathbb{R}^n$ and consider the three differential operators defined by

$$\begin{aligned} D_1 f &= \partial^2 f/\partial x_1^2 + \cdots + \partial^2 f/\partial x_n^2 , \\ D_2 f &= x_1 \partial f/\partial x_1 + \cdots + x_n \partial f/\partial x_n + (n/2) f , \\ D_3 f &= (x_1^2 + \cdots + x_n^2) f . \end{aligned} \tag{5}$$

Here D_1 is the Laplace operator on $\mathbb{R}^n$. Except for the addition of $n/2$, D_2 is the Euler degree operator. Finally, D_3 is just multiplication by the square of the distance to the origin. They satisfy the commutation relations

$$[D_1, D_2] = 2D_1 , \qquad [D_1, D_3] = -4D_2 , \qquad [D_2, D_3] = 2D_3 . \tag{6}$$

(The constant in D_2 has been added to simplify the formula for $[D_1, D_3]$.)

One reason to study these three operators together is that they all commute with the action of the orthogonal group $O(n)$ on functions; in fact, any polynomial coefficient differential operator commuting with $O(n)$ is in the algebra generated by D_1, D_2, and D_3. Another is the technical importance of the operators individually; analytically interesting properties can be expressed in terms of linear algebra and these operators. For example, a distribution is killed by a power of D_3 if and only if it is supported at the origin. Similarly, a distribution is an eigenvector for D_2 with eigenvalue λ if and only if it is homogeneous of degree $\lambda - n/2$. The Laplace operator D_1 is, of course, one of the most interesting differential operators in mathematics; as a random example, recall that a real-valued function on $\mathbb{R}^2 \simeq \mathbb{C}$ is killed by D_1 if and only if it is the real part of a holomorphic function.

The general philosophy of abstract harmonic analysis now says that to study the actions of D_1, D_2, and D_3 on functions, one should begin by studying an abstract vector space V endowed with three linear transformations A_1, A_2, and A_3, subject to the commutation relations

$$[A_1, A_2] = 2A_1, \qquad [A_1, A_3] = -4A_2, \qquad [A_2, A_3] = 2A_3. \tag{7}$$

The simplest nonzero linear transformations with these properties are the 2×2 matrices

$$A_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

These three matrices are a basis of the 2×2 matrices of trace zero: the Lie algebra $\mathfrak{sl}(2)$. The study of the equations (7) is therefore precisely the representation theory of $\mathfrak{sl}(2)$. For applications to differential operator representations, it is important not to assume that the vector space V is finite dimensional. (That special case is nevertheless of great interest in its own right; among other things, it is the key to the structure of semisimple Lie algebras. An excellent account of it is in [4].)

There are two basic ideas involved in analyzing (7). To understand the first, suppose that $v \in V$ is an eigenvector of A_2 of eigenvalue λ. Then $A_1 v$ turns out to be an eigenvector for A_2 of eigenvalue $\lambda - 2$. To see this, apply the commutation relation $(A_1A_2 - A_2A_1) = 2A_1$ to v and use the fact that $A_2 v = \lambda v$. The conclusion is that

$$\lambda A_1 v - A_2 A_1 v = 2 A_1 v.$$

This can be rearranged to

$$A_2(A_1 v) = (\lambda - 2) A_1 v,$$

which is the claim. A similar argument shows that $A_3 v$ is an eigenvector for A_2 of eigenvalue $\lambda + 2$. Now write $V_\lambda \subset V$ for the λ eigenspace of A_2. The actions of A_1 and A_3 on these subspaces are indicated by

$$\cdots V_{\lambda-2} \underset{A_3}{\overset{A_1}{\leftrightarrows}} V_\lambda \underset{A_3}{\overset{A_1}{\leftrightarrows}} V_{\lambda+2} \cdots \tag{9}$$

This diagram is the first main idea in the representation theory of $\mathfrak{sl}(2)$.

There is a technical difficulty that deserves mention here. It can easily happen that in an infinite-dimensional representation V, A_2 has no eigenvectors at all.

In this case all the spaces appearing in (9) are zero, and we have learned nothing about V. In our original example (5), eigenvectors of D_2 are homogeneous functions. If we want to study the action of the D_i on polynomials, it is therefore quite natural to look at eigenspaces of D_2 (and quite obvious that D_1 and D_3 lower and raise homogeneity degree by two). In other contexts homogeneous functions are not so natural, and other methods are needed to get something like (9). Consider, for example, the differential operators (5) acting on $L^2(\mathbb{R}^n)$. A homogeneous function in L^2 is zero, so D_2 has no eigenvectors in L^2. On the other hand, there is an orthonormal basis of L^2 consisting of eigenvectors of $D_1 - D_3$ (the Hermite functions). It turns out that $A_1 - A_3$ is conjugate to $2iA_2$ by an automorphism of the Lie algebra $\mathfrak{sl}(2)$ (the Cayley transform). Consequently there is an analogue of (9) for eigenspaces of $A_1 - A_3$, and this can be applied to $L^2(\mathbb{R}^n)$.

To analyze (9) further, we need to understand how the composition A_1A_3 acts on the eigenspace V_λ. For that purpose, consider the linear transformation

$$\Omega = A_2^2 + A_1A_3 + 2A_2. \tag{10}$$

This is called the *Casimir operator* of the representation V. Using the commutation relations (7), one can check easily that Ω commutes with all the A_i. It follows that the A_i preserve each eigenspace of Ω. Now we are interested in the simplest possible models of (7); an eigenspace of Ω would be a simpler one inside V. It is therefore interesting to consider the case when Ω is a multiple of the identity. (For the reader who finds this abstract argument unconvincing, here is a more concrete one. Write $D_\Omega = D_2^2 + D_1D_3 + 2D_2$ for the Casimir operator of the differential operator representation (5). This is a second-order differential operator on $\mathbb{R}^n$, which is not very difficult to write explicitly. If f is any smooth function on $\mathbb{R}^n$ transforming according to an irreducible representation of the orthogonal group $O(n)$, then f is necessarily an eigenfunction of D_Ω. This is proved in §III.2.3 of Howe and Tan's book.)

Proposition. *Suppose $(V, \{A_i\})$ is a representation of $\mathfrak{sl}(2)$. Then A_1 and A_3 act on the eigenspaces V_λ of A_2 as indicated in* (9). *If, in addition, the Casimir operator Ω is a scalar cI, then A_1A_3 acts on V_λ by the scalar $c - \lambda^2 - 2\lambda$.*

Corollary. *Suppose $(V, \{A_i\})$ is a representation of $\mathfrak{sl}(2)$ and $v_0 \in V_\lambda$ is an eigenvector of A_2. Assume also v_0 is an eigenvector for the Casimir operator Ω with eigenvalue c. Define*

$$v_n = \begin{cases} A_3^n v_0 & \text{if } n \geq 0, \\ A_1^{-n} v_0 & \text{if } n \leq 0. \end{cases}$$

Then $v_n \in V_{\lambda+2n}$ is an eigenvector for Ω of eigenvalue c and

$$A_1 v_n = \begin{cases} v_{n-1} & \text{if } n \leq 0, \\ (c - (\lambda + 2n - 2)^2 - 2(\lambda + 2n - 2))v_{n-1} & \text{if } n > 0; \end{cases}$$

$$A_2 v_n = (\lambda + 2n)v_n;$$

$$A_3 v_n = \begin{cases} v_{n+1} & \text{if } n \geq 0, \\ (c - (\lambda + 2n + 2)^2 - 2(\lambda + 2n + 2))v_{n+1} & \text{if } n < 0. \end{cases}$$

The last assertion of the proposition is immediate from (10); it is the second main idea in the representation theory of $\mathfrak{sl}(2)$. The corollary is a formal consequence. Notice that it comes close to describing the linear transformations A_i as explicit matrices in appropriate bases.

Armed with this information about possible solutions of the commutation relations (7), one can finally return to the differential operators (5) and say something about differential equations on $\mathbb{R}^n$. One gets first of all a very complete picture of the action of the D_i on polynomials; this is the classical theory of spherical harmonics, so called because it leads easily to the spectral decomposition of the Laplace operator on the $(n-1)$-dimensional sphere. Taking $L^2(\mathbb{R}^n)$ as the basic vector space leads first to information about the Hermite functions and then (less obviously) to the Fourier transform; indeed, the Fourier transform appears as an operator in a group representation whose differential is the Lie algebra representation we have been considering.

The development outlined above occupies three chapters—about the first half—of the book by Howe and Tan. Chapter I begins with a review of the theory of Lie groups and Lie algebras, continues with a summary of distribution theory, the Fourier transform, and the Schwartz space on $\mathbb{R}^n$, and concludes with a discussion of the spectrum of a Banach space representation of $\mathbb{R}^n$. (This is a generalization of the notion of the spectrum of a selfadjoint operator on a Hilbert space; it arises naturally in trying to make precise the idea mentioned above that commuting families of linear transformations ought to be understood in terms of simultaneous eigenvectors.) Chapter II is the technical heart of the book. It consists in large part of a refined version of the corollary stated above, describing possible realizations of the commutation relations (7). Chapter III applies these Lie algebra results to the representation theory of the group $SL(2, \mathbb{R})$ and its coverings. This leads to the (now classical) Bargmann classification of the irreducible unitary representations. Representations like (5) are also studied in detail, along with their analogues for the indefinite Laplacians $\sum_{i=1}^{p} \partial^2/\partial x_i^2 - \sum_{j=p+1}^{n} \partial^2/\partial x_j^2$.

In Chapter IV the deeper analytic applications begin. Some, like the discussion of fundamental solutions of the Laplacian, will be very familiar to classical analysts, but even in these cases the authors have a fresh perspective. Others are taken from Harish-Chandra's analysis on semisimple groups and are (unfortunately) almost unknown outside representation theory. Here is an example. Make the unitary group $U(n)$ act on the vector space $\mathfrak{g}$ of $n \times n$ skew-Hermitian matrices (its Lie algebra) by conjugation. (The results, appropriately recast, apply to any real reductive Lie group.) Write $\mathfrak{t}$ for the subspace of diagonal matrices (with purely imaginary entries). Any $U(n)$-invariant function f on $\mathfrak{g}$ is determined by its restriction $p(f)$ to $\mathfrak{t}$, since any skew-Hermitian matrix can be diagonalized by a unitary matrix. The Laplacian Δ on $\mathfrak{g}$ preserves $U(n)$-invariance. It follows more or less formally that there is a differential operator $r(\Delta)$ on $\mathfrak{t}$ (the *radial part* of Δ) with the property that

$$p(\Delta f) = r(\Delta)p(f).$$

(Actually $r(\Delta)$ is defined only on the open subset of $\mathfrak{t}$ consisting of matrices with distinct diagonal entries.) Harish-Chandra's restriction formula calculates the differential operator $r(\Delta)$; it is the conjugate of the Laplacian Δ_1 on $\mathfrak{t}$ by the discriminant function $D(t) = \prod_{i<j}(t_i - t_j)$. Putting in the definition of $r(\Delta)$, this becomes

$$D \cdot p(\Delta f) = \Delta_1(D \cdot p(f)).$$

A trivial formula of the same type applies to the zeroth-order operator of multiplication by the square of the distance to the origin. At this point we can bring $\mathfrak{sl}(2)$ into the problem. Recall that the Fourier transform appears in a group representation generated by these two differential operators. One can deduce a formula for the Fourier transform of a $U(n)$-invariant tempered function f on $\mathfrak{g}$ in terms of the Fourier transform of $p(f)$:

$$(11) \qquad \widehat{D \cdot p(f)} = i^{n(n-1)/2} D \cdot p(\hat{f})).$$

(Incidentally, one could try to write an analogous formula for the Fourier transform of rotationally invariant functions on $\mathbb{R}^n$ in terms of their restriction to $\mathbb{R}$. As is well known to classical analysts, there is no such formula except for $n = 1$ and $n = 3$. A typical feature of the book of Howe and Tan is a clear and simple representation-theoretic explanation of this fact.)

Harish-Chandra's restriction formula (11) is amazingly powerful. The reason is that the Fourier transform on all of $\mathfrak{g}$, even on $U(n)$-invariant functions, reflects the non-abelian nature of $U(n)$. For example, the Fourier transform of an invariant measure on a conjugacy class of skew-Hermitian matrices is more or less the character of an irreducible representation of $U(n)$, lifted to the Lie algebra. The restriction formula makes it possible to calculate these noncommutative things using classical abelian harmonic analysis on $\mathfrak{t}$. Such classical results as Weyl's character and dimension formulas follow fairly easily; this is a first step on the road leading to Harish-Chandra's Plancherel formula for semisimple groups. Staying within the setting of compact groups, one can also use (11) to study restriction of representations geometrically (as in [3]); here there is undoubtedly much more still to be done.

Chapter V is a brief excursion into representation theory for other groups, leading to Moore's ergodic theorem. The idea is that a noncompact semisimple Lie group G has a large supply of subgroups isomorphic to $SL(2, \mathbb{R})$. These subgroups make it possible to translate the very detailed information about unitary representations of $SL(2, \mathbb{R})$ collected in Chapter III into crude but nontrivial statements about unitary representations of G. These in turn can be translated (by ergodic theory) into statements about the action of G on a homogeneous space with a finite invariant measure.

The usually lucid exposition is marred by a few small mistakes and obscurities. The definition of Haar measure omits the requirement that compact sets have finite measure (perhaps because the definition of a measure inadvertently excludes sets of infinite measure). The formulation of the abstract Peter-Weyl decomposition for representations of compact groups is distressingly vague and omits any completeness hypothesis on the representation space. The sketch of the omitted proof that every unitary Harish-Chandra module for $SL(2, \mathbb{R})$ comes from a unitary representation appears to be incomplete. The analysis of invariant eigendistributions on $\mathfrak{sl}(2, \mathbb{R})$ is correct only for nonzero eigenvalues; a very careful reader might notice this assumption in the text at the top of page 185, but it is absent in the formulation of the theorem at the bottom of page 186. An erratum sheet deals with a more serious error in the description of $O(p, q)$-invariant distributions supported on the light cone.

Despite these minor problems, any graduate student with some understanding of manifolds and measure theory should be able to read this book. The most difficult sections require elementary hard work more often then deeper mathematics. The exercises are many and wonderful, leading the reader through dozens of interesting examples, omitted proofs, and explicit calculations.

If any mathematician *can* read it, who should? The authors suggest in the introduction that their book might serve as an introduction to the weightier tomes [5, 6], for a student studying semisimple harmonic analysis. This is certainly a possibility; but Knapp's book especially is already quite accessible, and many of the wonderfully particular things about $SL(2)$ discussed by Howe and Tan are not really necessary or helpful for understanding the general theory. (To be fair, the authors might argue that this reveals flaws in the general theory.)

But the authors also speak of their book as "a day hike to a nearby waterfall", and this is a better guide to who should read it. Every chapter is full of beautiful mathematics that is not as well known as it deserves to be. If you like waterfalls, come and have a look.

References

1. J. Björk, *Rings of differential operators*, North-Holland, Amsterdam, Oxford, and New York, 1979.
2. A. Borel et al., *Algebraic D-modules*, Perspec. Math., vol. 2, Academic Press, Boston, MA, 1987.
3. G. Heckman, *Projections of orbits and asymptotic behaviour of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1982), 333–356.
4. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1972.
5. A. Knapp, *Representation theory of real semisimple groups: an overview based on examples*, Princeton Univ. Press, Princeton, NJ, 1986.
6. N. Wallach, *Real reductive groups*, vol. I, Academic Press, San Diego, CA, 1988.

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Classical recursion theory, by P. Odifreddi. North-Holland, Amsterdam, 1992, xvii+668 pp., $63.00. ISBN 0-444-89483-7

Recursion theory as we know it today was born in the head of Alonzo Church one day in 1934. Church and his students, in an (eventually unsuccessful) effort to axiomatize the notion of a function, had arrived at the notion of a lambda definable function. A rather trivial consequence of the definition was that every lambda definable function was computable; i.e., the value of the function could be computed on a computer using the arguments as input. (Everyone who has programmed realizes that more powerful computers do not compute more functions; they simply compute the old functions faster and more easily.) Church had the radical thought that all computable functions were lambda definable. This was a remarkable foresight, since at that moment it had not yet been proved that the function $f(x) = x - 1$ was lambda definable. His student