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MÖBIUS INVARIANCE OF KNOT ENERGY

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ABSTRACT. A physically natural potential energy for simple closed curves in $\mathbf{R}^3$ is shown to be invariant under Möbius transformations. This leads to the rapid resolution of several open problems: round circles are precisely the absolute minima for energy; there is a minimum energy threshold below which knotting cannot occur; minimizers within prime knot types exist and are regular. Finally, the number of knot types with energy less than any constant M is estimated.

Consider a rectifiable curve $\gamma(u)$ in the Euclidean 3-space $\mathbf{R}^3$, where u belongs to $\mathbf{R}^1$ or S^1. Define its energy by

$$E(\gamma) = \iint \left\{ \frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{D(\gamma(u), \gamma(v))^2} \right\} |\dot{\gamma}(u)||\dot{\gamma}(v)| \, du \, dv ,$$

where $D(\gamma(u), \gamma(v))$ is the shortest arc distance between $\gamma(u)$ and $\gamma(v)$ on the curve. The second term of the integrand is called a regularization (see [O1–O3, FH]). It is easy to see that $E(\gamma)$ is independent of parametrization and is unchanged if γ is changed by a similarity of $\mathbf{R}^3$.

Recall that the Möbius transformations of the 3-sphere $= \mathbf{R}^3 \cup \infty$ are the ten-dimensional group of angle-preserving diffeomorphisms generated by inversion in 2-spheres.

The central fact of this announcement is:

Möbius Invariant Property. *Let γ be a closed curve in $\mathbf{R}^3$. If T is a Möbius transformation of $\mathbf{R}^3 \cup \infty$ and $T(\gamma)$ is contained in $\mathbf{R}^3$, then $E(T(\gamma)) = E(\gamma)$. If $T(\gamma)$ passes through ∞, the integral satisfies $E(T(\gamma)) = E(\gamma) - 4$.*

This simple fact (proved below), combined with earlier results proved in [FH], allows the rapid resolution of several open problems.

Theorem A. *Among all rectifiable loops γ: $S^1 \to \mathbf{R}^3$, round circles have the least energy (E (round circle) $= 4$) and any γ of least energy parameterizes a round circle.*

Theorem B. *If K is a smooth prime (not a connected sum) knot, then there exists a simple closed rectifiable γ_K of knot type K with $E(\gamma_K) \leq E(\gamma)$ for all rectifiable loops γ which are topologically ambient isotopic to K.*

Theorem C. *Any minimizer γ_K, as above, will enjoy some regularity. With an arc length parametrization, γ_K will be in $C^{1,1}$.*

Several results of [FH] can be improved quantitatively.

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Theorem D. *If γ is topologically tame, let $c([\gamma])$ denote the (topological) crossing number of the knot type. We have*

$$2\pi c([\gamma]) + 4 \leq E(\gamma).$$

(*It was proved in* [FH] *that finite energy implies tame.*)

Since an essential knot must have three or more crossings, we obtain the following

Corollary. *Any rectifiable loop with energy less than $6\pi + 4 \approx 22.84954$ is unknotted.*

Computer experiments of [A] as reported in [O3] and independently by the first author yield an essential knot (a trefoil) with energy ≈ 74.

It may be estimated [S,T,W] that the number $K(n)$ of distinct knots of at most n crossings satisfies

$$2^n \leq K(n) \leq 2 \cdot 24^n.$$

Hence the number of knot types with representatives below a given energy threshold can also be bounded by an exponential.

Theorem E. *The number $K_e(M)$ of isomorphism classes of knots which have representatives of energy less than or equal to M is bounded by $2(24^{-4/2\pi})(24^{1/2\pi})^M \approx (0.264)(1.658)^M$. In particular, only finitely many knot types occur below any finite energy threshold.*

Note that there are competing candidates for the exponent $= -2$ in the definition of E; for example, the Newtonian potential in $\mathbf{R}^3$ has exponent $= -1$. When the exponent is strictly larger than -3, finite values are obtained for smooth simple loops. Exponents smaller or equal to -2 yield energies which blow up as a simple loop γ begins to acquire a double point, thus creating an infinite energy barrier to a change of topology. Such a barrier would not exist for the Newtonian potential. We refer to [O1–O3] for detailed discussions. Similarity and Möbius invariance are, of course, special to the exponent -2.

Proof of Theorem A. Let T be a Möbius transformation sending a point of γ to infinity. The energy $E(T(\gamma)) \geq 0$ with equality holding iff $T(\gamma)$ is a straight line. Apply the Möbius invariant property to complete the proof. $\square$

Proof of Theorem B. In [FH] it is shown that for prime knot types K minimizers exist in the class of properly embedded rectifiable lines whose completion in $\mathbf{R}^3 \cup \infty$ represent K. According to the Möbius Invariance Property, such lines may be moved to a closed minimizer by any Möbius transformation T which moves the completed line off infinity. $\square$

Sketch of Proof of Theorem C. Let γ_K be a closed minimizer in knot type K. An inversion argument shows that, for sufficiently small $\varepsilon > 0$, if γ_K meets a closed ball of radius ε, B_ε, only in its boundary S_ε, then $\gamma_K \cap S_\varepsilon$ consists of (at most) one point. The idea is that if $\gamma_K \cap S_\varepsilon$ is disconnected, inverting an arc of $\gamma_K \backslash S_\varepsilon$ into B_ε will lower energy while preserving the knot type. Thus there is a continuous projection from the ε-neighborhood of γ_K to γ_K given by "closest point" $\pi\colon \mathcal{N}_\varepsilon(\gamma_K) \to \gamma_K$. We prove that the fibers $\pi^{-1}(pt)$ are all geometric planar disks of radius ε. The disjointness of these "normal" fibers to distance ε is equivalent to the existence of a continuously turning tangent to γ_k whose generalized derivative is in L^∞. $\square$

A detailed proof of Theorem C will appear elsewhere.

Proof of Theorem D. Theorem 2.5 of [FH] gives the inequality

$$c([\gamma]) \le c(\gamma) \le E(\gamma)/2\pi$$

for proper rectifiable lines. According to the Möbius Invariance Property, the energy will increase by exactly 4 if a Möbius transformation is used to move the line off infinity and into closed position. $\square$

Proof of Möbius Invariance Property. It is sufficient to consider how I, an inversion in a sphere, transforms energy. Let u be the arc length parameter of a rectifiable closed curve γ, $u \in \mathbf{R}/l\mathbf{Z}$. Let

$$E_\varepsilon(\gamma) = \iint_{|u-v|\ge\varepsilon} \left(\frac{1}{|\gamma(u)-\gamma(v)|^2} - \frac{1}{(D(\gamma(u), \gamma(v)))^2}\right) du\, dv \tag{1}$$

and

$$E_\varepsilon(I\circ\gamma) = \iint_{|u-v|\ge\varepsilon} \left(\frac{1}{|I\circ\gamma(u)-I\circ\gamma(v)|^2} - \frac{1}{(D(I\circ\gamma(u), I\circ\gamma(v)))^2}\right) \times \|I'(\gamma(u))\|\cdot\|I'(\gamma(v))\|\, du\, dv. \tag{2}$$

Clearly $E(\gamma) = \lim_{\varepsilon\to0} E_\varepsilon(\gamma)$ and $E(I\circ\gamma) = \lim_{\varepsilon\to0} E_\varepsilon(I\circ\gamma)$.

It is a short calculation (using the law of cosines) that the first terms transform correctly, i.e.,

$$\frac{\|I'(\gamma(u))\|\cdot\|I'(\gamma(v))\|}{|I(\gamma(u))-I(\gamma(v))|^2} = \frac{1}{|\gamma(u)-\gamma(v)|^2}.$$

Since u is arclength for γ, the regularization term of (1) is the elementary integral

$$\int_{u=0}^{l}\left[2\int_{v=\varepsilon}^{l/2}\frac{1}{v^2}\,dv\right]du = 4 - \frac{2l}{\varepsilon}. \tag{3}$$

Let s be an arclength parameter for $I\circ\gamma$. Then $ds(u)/du = \|I'(\gamma(u))\|$ where $\|I'(\gamma(u))\| = f(u)$ denotes the linear expansion factor of I'. Since $\gamma(u)$ is a lipschitz function and I' is smooth, $f(u)$ is lipschitz, hence, it has a generalized derivative $f'(u)\in L^\infty$.

(4)

$$\begin{aligned}\text{regularization (2)} &= \int_{u\in\mathbf{R}/l\mathbf{Z}}\left[\int_{|v-u|\ge\varepsilon}\frac{|(I\circ\gamma)'(v)|\,dv}{D(I\circ\gamma(u), I\circ\gamma(v))^2}\right]|(I\circ\gamma)'(u)|\,du\\ &= \int_{\mathbf{R}/l\mathbf{Z}}\left[\frac{4}{L} - \frac{1}{\varepsilon_+} - \frac{1}{\varepsilon_-}\right]ds,\end{aligned}$$

where $L = \text{Length}(I(\gamma))$ and

$$\begin{aligned}\varepsilon_+ = \varepsilon_+(u) &= D((I\circ\gamma)(u), (I\circ\gamma)(u+\varepsilon)) = s(u+\varepsilon) - s(u)\\ &= \int_u^{u+\varepsilon} f(w)\,dw = f(u)\varepsilon + \varepsilon^2\int_0^1 (1-t)f'(u+\varepsilon t)\,dt\end{aligned}$$

and

$$\varepsilon_- = \varepsilon_-(u) = D((I \circ \gamma)(u - \varepsilon), (I \circ \gamma)(u)) = f(u)\varepsilon - \varepsilon^2 \int_0^1 (1-t) f'(u - \varepsilon t)\, dt.$$

Since $|f'(w)|$ is uniformly bounded, we have

$$\begin{aligned}
\frac{1}{\varepsilon_+} &= \frac{1}{f(u)\varepsilon} \left[\frac{1}{1 + (\varepsilon/f(u)) \int_0^1 (1-t) f'(u + \varepsilon t)\, dt} \right] \\
&= \frac{1}{f(u)\varepsilon} \left[1 - \frac{\varepsilon}{f(u)} \int_0^1 (1-t) f'(u + \varepsilon t)\, dt + \mathcal{O}(\varepsilon^2) \right] \\
&= \frac{1}{f(u)\varepsilon} - \frac{1}{f(u)^2} \int_0^1 (1-t) f'(u + \varepsilon t)\, dt + \mathcal{O}(\varepsilon).
\end{aligned}$$

Similarly,

$$\frac{1}{\varepsilon_-} = \frac{1}{f(u)\varepsilon} + \frac{1}{f(u)^2} \int_0^1 (1-t) f'(u - \varepsilon t)\, dt + \mathcal{O}(\varepsilon).$$

Then by (4)

(5)

$$\begin{aligned}
\text{regularization (2)} &= 4 - \int_{\mathbf{R}/l\mathbf{Z}} \frac{2}{\varepsilon}\, du \\
&\quad + \iint_{\mathbf{R}/l\mathbf{Z} \times [0,1]} \frac{(1-t)}{f(u)} [f'(u + \varepsilon t) - f'(u - \varepsilon t)]\, du\, dt + \mathcal{O}(\varepsilon) \\
&= 4 - \frac{2l}{\varepsilon} + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Comparing (3) and (5), we get

$$E_\varepsilon(\gamma) - E_\varepsilon(I \circ \gamma) = \mathcal{O}(\varepsilon);$$

hence, $E(\gamma) = E(I \circ \gamma)$.

For the second assertion, let I send a point of γ to infinity. In this case $L = \infty$ and, thus, the constant term 4 in (5) disappears. □

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