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RELATIVE K-CYCLES AND ELLIPTIC BOUNDARY CONDITIONS

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Dedicated to Professor Zejian Jiang on his seventieth birthday

ABSTRACT. In this paper, we discuss the following conjecture raised by Baum-Douglas: For any first-order elliptic differential operator D on smooth manifold M with boundary ∂M, D possesses an elliptic boundary condition if and only if $\partial[D] = 0$ in $K_1(\partial M)$, where $[D]$ is the relative K-cycle in $K_0(M, \partial M)$ corresponding to D. We prove the "if" part of this conjecture for $\dim(M) \neq$ 4, 5, 6, 7 and the "only if" part of the conjecture for arbitrary dimension.

First we fix some notation. M is a compact oriented smooth manifold with smooth boundary ∂M. We always suppose that M is embedded in some compact smooth manifold $\widetilde{M}$ without boundary of the same dimension (e.g., $\widetilde{M}$ can be taken as double of M). We denote $\overset{\circ}{M} = M \setminus \partial M$. Furthermore, we assume that E_0 and E_1 (in fact, all the vector bundles in this paper) are smooth complex Hermitian vector bundles over M and that $D : C^\infty(E_0) \to C^\infty(E_1)$ is a first-order elliptic differential operator from smooth sections of E_0 to that of E_1. By $H^s(M, E_i)$ and $H^s(\partial M, E_i)$ we shall denote the Sobolev spaces of sections of E_i and $E_i|_{\partial M}$ with respect to fixed smooth measures on M and ∂M, respectively.

The elliptic boundary value problem (an elliptic operator with an elliptic boundary condition) has been studied for a long time. As noted in [1, 5, 6] and other references, there exist topological obstructions to impose an elliptic boundary condition on the above D. A fundamental problem is to find all such obstructions. Baum, Douglas, and Taylor constructed a relative K-cycle $[D] \in K_0(M, \partial M) \cong KK(C_0(\overset{\circ}{M}), \mathbb{C})$ (here $C_0(\overset{\circ}{M})$ is the algebra of continuous functions on M which vanish on ∂M) corresponding to D (see [2–4] for details). From the definition of relative K-homology group $K_0(M, \partial M)$ given by Baum, Douglas, and Taylor, the boundary map $\partial \colon K_0(M, \partial M) \longrightarrow K_1(\partial M)$ is very concrete [2–4]. Also Baum and Douglas conjectured that the only obstruction for D possessing elliptic boundary conditions is that $\partial[D] \neq 0$. More precisely, the following conjecture first appeared in [2] in a closely related form.

Conjecture. *There exist a vector bundle E_2 over ∂M and a zeroth-order pseudo-*

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differential operator B defined from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ such that

$$\begin{pmatrix} D \\ B\circ\gamma \end{pmatrix}: \quad H^1(M,E_0) \longrightarrow \begin{matrix} H^0(M,E_1) \\ \oplus \\ H^{1/2}(\partial M, E_2) \end{matrix}$$

is Fredholm if and only if $\partial[D]=0$ in $K_1(\partial M)$, where $\gamma : H^1(M,E_0) \longrightarrow H^{1/2}(\partial M, E_0)$ is the trace map.

Remark 1. Let D be as above with pincipal symbol $p(x,\xi)$. A zeroth-order pseudo-differential operator B with principal symbol $b(x,\xi)$ from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ is said to be elliptic to D (see [6], p. 233) if, for every $x \in \partial M$ and $\xi \in T^*_x(\partial M)$, the map $M^+_{x,\xi} \ni u \longrightarrow b(x,\xi)u(0) \in (E_2)_x$ is bijective, where $T^*_x(\partial M)$ and $(E_2)_x$ are the fibres at the point x of the cotangent bundle $T^*\partial M$ and the bundle E_2, and, furthermore, $M^+_{x,\xi}$ is the set of all $u \in C^\infty(\mathbb{R}, (E_0)_t)$ with $p(x, \xi - i\frac{d}{dt}\cdot n_x)u(t) = 0$ (n_x is the interior conormal vector of M at x) which are bounded on $\mathbb{R}_+$. If B is elliptic to D, then the above $\left(\begin{smallmatrix} D \\ B\circ\gamma \end{smallmatrix}\right)$ is Fredholm. Such a system $\left(\begin{smallmatrix} D \\ B\circ\gamma \end{smallmatrix}\right)$ is often called an elliptic boundary value problem or an elliptic operator with an elliptic boundary condition; meanwhile, D is also said to possess an elliptic boundary condition.

Remark 2. Although the above elliptic boundary condition is used in most references, the original form of the conjecture in [2] is in a slightly different form from the above. In [2] the operator for the boundary condition is of the form $\gamma \circ B$, where B is a zeroth-order pseudo-differential operator from E_0 to a smooth vector bundle over a neighborhood of M. The reason we use a slightly different form of the conjecture is as follows: for general zeroth-order pseudo-differential operator B defined on $\widetilde{M}$, there is no canonical way to restrict B to M as an operator $B_M : H^s(M) \longrightarrow H^s(M)$ when $s > 0$. So one needs to put some restriction on B. One of the natural restrictions is that B has the transmission property with respect to ∂M (see [5]). We also prove our theorem for this kind of boundary condition (see Theorem 1). It must be pointed out that the existence of a boundary condition of type $B \circ \gamma$ implies the existence of that of type $\gamma \circ B$.

In this paper, we prove the "only if" part of the conjecture which can be thought of as a generalization of Corollary 4.2 in [4] (there B is a differential operator). Conversely, we prove that if $\dim(M) \neq 4,5,6,7$ and $\partial[D]=0$, then D possesses an elliptic boundary condition as in Remark 1. Hence the "if" part of the conjecture has been proved for $\dim(M) \neq 4,5,6,7$. The cases of $\dim(M)$ being equal to $4,5,6$, or 7 are still open, but we prove a theorem which can be thought of as the "if" part of the conjecture in the sense of stablization in K-homology group for arbitrary dimension. Our results will be useful for constructing absolute K-cycles in $K_0(M)$ which are preimages of $[D] \in K_0(M, \partial M)$ under the canonical map from $K_0(M)$ to $K_0(M, \partial M)$ when $\partial[D]=0$.

Our main results are the following:

Theorem 1. ("*only if* " *part*) $\partial[D]=0$ *if one of the following is true*:

(i) *There exist a smooth vector bundle E_2 over ∂M and a zeroth-order pseudo-differential operator B from $E_0|_{\partial M}$ to E_2 such that $\left(\begin{smallmatrix} D \\ B\circ\gamma \end{smallmatrix}\right)$ in the conjecture is Fredholm.*

(ii) *There exist a bundle* E_2 *over a neighborhood of* M *in* $\widetilde{M}$ *and a zeroth-order pseudo-differential operatorB with transmission property with respect to* ∂M *from* E_0 *to* E_2 *such that*

$$\begin{pmatrix} D \\ \gamma \circ B \end{pmatrix} : \quad H^1(M, E_0) \longrightarrow \begin{matrix} H^0(M, E_1) \\ \oplus \\ H^{1/2}(\partial M, E_2) \end{matrix}$$

is Fredholm.

Theorem 2. ("*if* " *part*) *If* $\partial[D] = 0$*, then there exists a first-order elliptic differential operator* D_1 *acting on smooth vector bundles over* M *with* $[D_1] = 0$ *in* $K_0(M, \partial M)$ *such that* $D \oplus D_1$ *possesses an elliptic boundary condition as in Remark* 1*, and, furthermore, if* $\dim(M) \neq 4, 5, 6, 7$*, then* D *itself possesses an elliptic boundary condition.*

The main idea of the proof of Theorem 1 is to construct an intertwining between $\partial[D]$ and a trivial element in $K_1(\partial M)$. In the proof, we use Calderon projection, functional calculus of pseudo-differential operators (including Boutet de Monvel type operators), and the techniques in the proof of Proposition 4.5 of [4].

The proof of Theorem 2 makes use of two key lemmas (see below).

Let $ST^*\partial M$ be the unit sphere bundle of $T^*\partial M$ over ∂M and $\pi \colon ST^*\partial M \longrightarrow \partial M$ be the canonical projection map. Let $\widetilde{E}_0 = \pi^*(E_0|_{\partial M})$ be the bundle over $ST^*\partial M$. We write the principal symbol of D, in a coordinate neighborhood U of $x \in \partial M$, as

$$p(x, x_n, \xi, \xi_n) = \sum_{j=1}^{n-1} p_j(x, x_n)\xi_j + p_n(x, x_n)\xi_n,$$

where x_n is the coordinate for the normal direction of ∂M. We define

$$\tau(x, \xi) = ip_n^{-1}(x, 0) \sum_{j=1}^{n-1} p_j(x, 0)\xi_j$$

for $x \in \partial M$ and $\xi \in ST^*\partial M$. Then $\tau(x, \xi)$ is a map from a fibre of E_0 into itself and has no purely imaginary eigenvalue. Let $V_\pm$ be the subbundle of $\widetilde{E}_0$ over $ST^*\partial M$ corresponding to the span of the generalized eigenvectors of $\tau(x, \xi)$ corresponding to the eigenvalues with positive/negative real parts.

Lemma 1. $\partial[D] = 0$ *if and only if* $[V_+] \in \pi^* K^0(\partial M) \subset K^0(ST^*\partial M)$.

Lemma 2. *If* E_0 *and* E_1 *are vector bundles over* M *which allow a first-order elliptic differential operator* D *to act from one to the other, and if* $\dim(M) = n$*, then*

(i) $f\dim(E_0) = f\dim E_1 \geq 2^{[(n-1)/2]}$;

(ii) $f\dim(E_0) = f\dim E_1 \geq 2^{[(n-1)/2]+1}$ *provided* n *is even and* $\partial[D] = 0$*, where* $f\dim$ *denotes dimension of each fibre of the vector bundles.*

The Proof of Theorem 2. By Lemma 1, if $\partial[D]=0$, one has

$$[V_+]\in\pi^*K^0(\partial M)\subset K^0(ST^*\partial M).$$

By Lemma 2, $f\dim(V_+)=\frac{f\dim(E_0)}{2}\geq 2^{[n/2]-1}$. Therefore, $f\dim(V_+)\geq n-1>\frac{\dim(ST^*\partial M)}{2}=\frac{2n-3}{2}$, whenever $\dim(M)\geq 8$. Hence there exists a complex vector bundle E_2 over ∂M, such that $V_+\cong\pi^*E_2$. This is also true for $\dim(M)\leq 3$, since the collection of complex vector bundles over $ST^*\partial M$ ($\dim(ST^*\partial M)\leq 3$) has property of cancellation.

Let ψ be the bundle isomorphism

$$\begin{array}{ccc} V_+ & \xrightarrow{\psi} & \pi^*E_2 \\ \downarrow & & \downarrow \\ ST^*\partial M & \longrightarrow & ST^*\partial M \end{array}$$

For any $(x,\xi)\in ST^*\partial M$, let $b(x,\xi)$ be the bundle map defined by

$$\begin{array}{ccccc} E_0 & \xrightarrow{\text{project to}} & V_+ & \xrightarrow{\psi} & E_2 \\ \downarrow & & & & \downarrow \\ ST^*\partial M & \longrightarrow & ST^*\partial M & \longrightarrow & ST^*\partial M \end{array}$$

Furthermore, let $B: E_0|_{\partial M}\longrightarrow E_2$ be the zeroth-order pseudo-differential operator with symbol $b(x,\xi)$. It follows that B is elliptic to D (see [6]). □

Example 1. If M is a *spin*^{c} manifold with smooth boundary and D is the Dirac operator over M, then it is computed in [4] that $\partial[D]\neq 0$. Hence D possesses no elliptic boundary condition (even possesses no boundary condition as in the conjecture).

Example 2. For any D, let D^* be the formal adjoint of D. It is easy to prove that $[D]=-[D^*]$ in $K_0(M,\partial M)$; hence, $\partial[D\oplus D^*]=0$. It follows that $D\oplus D^*$ possesses an elliptic boundary condition provided $\dim(M)\neq 4$ or 6. (It should be noted that we only need to exclude the manifolds with dimension 4 and 6 here, since the dimension of the bundle on which $D\oplus D^*$ acts is twice the dimension of the bundle on which D acts.)

The details of the proofs will appear elsewhere.

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References

1. M. Atiyah, V. Patodi, and I. Singer, *Spectral asymmetry and riemannian geometry.* I, II, III, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69; **78** (1975), 405–632; **79** (1976), 71–99.
2. P. Baum and R. Douglas, *Index theory, bordism, and K-homology,* Operator Algebras and K-Theory (R. G. Douglas and C. Schochet, eds.), Contemp. Math., vol. 10, Amer. Math. Soc., Providence, RI, 1982, pp. 1–31.
3. ______, *Relative K-homology and C^*-algebra,* manuscript.
4. P. Baum, R. Douglas, and M. Taylor, *Cycles and relative cycles in analytic K-homology,* J. Differential Geom. **30** (1989), 761–804.
5. L. Boutet de Monvel, *Boundary problems for pseudodifferential operators,* Acta Math. **126** (1971), 11–51.
6. L. Hömander, *The analysis of linear partial differential operators*. III, Springer, New York, 1985.

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