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BOOK REVIEW

Banach lattices, by Peter Meyer-Nieberg. Springer-Verlag, New York, 1991, 395 pp., $49.95. ISBN 3-540-54201-9

Normed vector spaces were introduced and studied in a systematic manner for the first time by Banach in the early 1920s and the theory of normed spaces was established as part of functional analysis with the publication of Banach's classic monograph [5] in 1932. From then on, the theory of Banach spaces experienced a rapid development that in its ranks of researchers included several prominent mathematicians.

Most of the early research on Banach spaces was focused on algebraic and topological properties and little attention was paid to order structures. F. Riesz in his address to the 1928 International Congress of Mathematicians [26, 27] in Bologna, Italy, proposed that parallel with the study of algebraic and topological properties of normed spaces a study of their order properties must also be undertaken. He thus laid down the beginning of the study of partially ordered vector spaces. Soon after, Kantorovich [15, 16] in the Soviet Union, Freudenthal [12] in Holland, and Birkhoff [6, 1940 Edition] in the United States, proposed the first axiomatic foundations of vector lattices (=Riesz spaces). Surprisingly, in the next fifty years, the paths of normed and partially ordered vector spaces were separated with few occasional crossings—the two fields were developed practically in their own separate ways.

The theory of partially ordered vector spaces was developed through the efforts of mathematicians from many countries. During the last sixty years, there were several monographs written on the subject, the first written by Nakano [23, 24]. The monographs by Peressini [25], Jameson [13], and the last part of Schaefer's book [28] were devoted entirely to order properties of vector spaces. The books by Luxemburg and Zaanen [21], Vulikh [31], and Kaplan [17] study the order structure of Riesz spaces while Fremlin [11], Wong and Ng [32], and the reviewer and Burkinshaw [2] study topological properties of ordered vector spaces. Extensive treatments on Banach lattices and positive operators can be found in books by Lacey [18], Lindenstrauss and Tzafriri [20], Schaefer [29], Schwarz [30], Zaanen [34], and the reviewer and Burkinshaw [3]. The monograph by Arendt et al. [4] deals with semigroups of positive operators.

The theory of Riesz spaces and positive operators has some immediate applications to a variety of fields in other disciplines. The following is a list of some areas where Banach lattices and positive operators play an important role:

- Nuclear-reactor theory,
- Population dynamics,

- Statistical decision theory,
- Economics,
- Finance,
- Game theory.

The nuclear-reactor theory can be modeled by a set of functional equations that are associated with kernel operators. When reactor problems are solved by the use of computers, one uses a "finite" model of the reactor. In such a model, the kernel operators are reduced to positive matrices and the "criticality problem" is reduced to finding the largest eigenvalue of a positive matrix. The Perron-Frobenius theory and the theory of positive operators are usually used in connection with this problem (see [7, 8]). Likewise, in population dynamics, positive operators appear naturally with models described by certain "growth" partial differential equations (see [9, 22] and the references therein for details).

Le Cam [19] has chosen to exposit the asymptotic theory of statistical decision theory within the framework of Riesz spaces. We quote from his Preface: "... we elected to work with vector lattices and positive linear operators instead of families of measures and Markov kernels. The main reason for this choice is that it allows one to work essentially as if all objects used were finite sets. The basic completeness results and the minimax theorem are always true, instead of being encumbered by unappetizing ad hoc assumptions....".

In economics, Banach lattices appear naturally within the context of general equilibrium theory. If $S(p)$ and $D(p)$ are the supply and demand functions as functions of the price vector p, then the basic question in general equilibrium theory is to give conditions on $S(\cdot)$ and $D(\cdot)$ that guarantee the existence of a price vector q such that $S(q) = D(q)$. Market economies allocate commodities by means of a system of prices. The duality between commodities and prices is expressed in theoretical terms as a dual pair $\langle X, X'\rangle$ of ordered vector spaces. The vector space X is the commodity space, the vector space X' is the price space, and the evaluation $\langle x, x'\rangle$ is interpreted as the value of the commodity bundle $x \in X$ at prices $x' \in X'$. All market activity in an economy can be described in terms of the structure of the dual pair $\langle X, X'\rangle$. When X is a Banach lattice and X' is its norm dual, then the rich topological and lattice structures of the spaces allow us to prove the existence of equilibrium prices. For details see [1].

A particularly natural application of the theory of Riesz spaces and Banach lattices in general equilibrium theory is as models for financial markets. In the abstract theory of finance (see Duffie [14, Chapter 31]) the space of financial assets is a subset of the space of real-valued functions on some set Ω. The set Ω is the set of possible states of the world tomorrow, $f \in R^{\Omega}$ is an asset where $f(\omega)$ is the asset's payoff if the state of the world tomorrow is ω. The natural operations for creating new assets are the vector space operations and the lattice operations defined pointwise. The importance of the lattice operations can be appreciated as follows. Suppose $s(\omega)$ is some risky asset, e.g., a stock, and $b(\omega)$ is the riskless asset, e.g., a bond, which pays one in every state of the world. A European call option on s is then $(s - kb)^+$ where k is some nonnegative real number, called the striking price. The interpretation of $(s - kb)^+$ is that the holder of the call option has the right to call away tomorrow the stock s for the striking price k. Clearly, an investor will only exercise his option if $s(\omega) > kb(\omega)$; hence, the payoff of the call option is $(s - kb) \vee 0$ or $(s - kb)^+$.

Banach lattices also arise naturally in economies and games with differential information. In particular, the random consumption set (or strategy set) of an agent is a set-valued function $X\colon \Omega \to E$, where (Ω, Σ, μ) is a probability space and E is a Banach lattice with order continuous norm. Since consumption bundles are assumed to be Bochner integrable selections from $X(\cdot)$, the set of all such consumption bundles is

$$L_X = \{x \in L_1(\mu, E) : x(\omega) \in X(\omega) \text{ for } \mu - \text{almost all } \omega\}.$$

Here $L_1(\mu, E)$ is the Banach space of all equivalence classes of all Bochner integrable functions normed by $\|x\| = \int_\Omega \|x(\omega)\|\, d\mu(\omega)$, which, as it turns out, is a Banach lattice with order continuous norm. Consequently, $L_1(\mu, E)$ has weakly compact order intervals, and from this it follows that L_X is a weakly compact set. Yannelis [33] has shown that the latter is of extreme importance in proving existence of equilibria for games and economies with differential information.

Now we come to the book under review. The monograph presents an extensive treatment of Banach lattices and positive operators acting between them. The author is one of the distinguished contributors to the subject—he belongs to Schaefer's German school of functional analysis.

Chapter 1 covers the basic properties of Riesz spaces, Banach lattices, and regular operators; order continuous operators, lattice homomorphisms, and extensions of positive operators are all studied in this chapter. Chapter 2 presents a comprehensive study of classical Banach lattices $C(K)$ and $L_p(\mu)$; disjoint sequences, order continuous norms, weak compactness, and reflexivity are also discussed here. In addition, in this chapter, the author studies Banach function spaces, characterizations of L_p-spaces, and cone p-absolutely summing operators and their factorizations.

Chapter 3 deals extensively with the fundamental properties of positive operators. Here the reader will find an extensive treatment of disjointness preserving operators, orthomorphisms (multiplication operators), and kernel operators—Bukhvalov's beautiful characterization of kernel operators is proven here. In particular, this chapter presents a systematic study of various compactness properties associated with operators, namely, compactness, weak compactness, Dunford-Pettis property, L- and M-weak compactness, o-weak compactness, etc. Several compactness properties of positive operators dominated by compact or weakly compact operators are established here. For instance, the Dodds-Fremlin theorem [10] (asserting that if E' and F have order continuous norms, then every positive operator from E into F which is dominated by a compact operator is itself compact) is proven in this chapter. Moreover, numerous factorization properties of positive operators are established.

Chapter 4 is entirely devoted to spectral properties of positive operators. Many of the results in this chapter have been obtained since 1985 but are included in a book for the first time; they include: irreducible operators (B. de Pagter's theorem on the positivity of the spectral radius of irreducible positive compact operators is proven here), measures of noncompactness, local spectral theory, the order spectrum, disjointness preserving operators. Lattice versions of the zero-two-law due to Zaharopol, Schaefer, Katznelson and Tzafriri are also established in this chapter; for instance, it is shown that if $T\colon E \to E$ is a positive contraction on a Dedekind complete Banach lattice, then either $\|T^n - T^{n+1}\|_r \equiv \||T^n - T^{n+1}|\| \to 0$ or else $\|T^n - T^{n+1}\|_r = 2$ for each n.

The final chapter (Chapter 5) is reserved for Banach space properties of Banach lattices. It studies Banach lattices with subspaces isomorphic to $C(\Delta)$, Grothendieck spaces, and the Radon-Nikodym property. In particular, the author presents a comprehensive study of the properties (V) and (V^*), operators not preserving subspaces isomorphic to l_1, and representable operators. These last two chapters contain a wealth of material most of which appears for the first time in book form.

The monograph is very well written and is certainly a welcome addition to the subject. It complements and supplements the existing books. Besides being a research monograph, the book can be used as a text for advanced undergraduate and graduate courses in functional analysis—the challenging exercises at the end of each section make the book suitable for a text. In addition, the book will be useful to researchers in mathematics and scholars in other disciplines outside mathematics. The reviewer highly recommends it to anyone interested in the "positivity and order" aspects of functional analysis.

My only criticism of the book is that the author does not always give enough credit to non-German mathematicians who contributed to the field. Although this is understandable, it cannot relieve the author from the moral responsibility and the obligation of reporting a historically accurate account of the contributors to the subject matter of the book.

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