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BOOK REVIEW

Two-dimensional geometric variational problems, by Jürgen Jost. Wiley, New York, 1991, x+234 pp. \$87.95, ISBN 0-471-92839-9

In the first chapter the author presents the most important and fascinating geometric variational problem, namely, to span a disc-type surface of minimal area into a given spatial Jordan curve Γ. This mathematical problem can be realized by soap films spanning into a given wire, initiated by experiments of the Belgian physicist Plateau. Then the general two-dimensional conformally invariant variational problem is considered, where the function $u \in H^{1,2}(\Omega, \mathbb{R}^d)$ with $d \geq 2$ renders the integral

$$I(u) := \frac{1}{2}\int_\Omega \{g_{ik}(u)\nabla u^i \nabla u^k + b_{ik}(u)\det(\nabla u^i, \nabla u^k)\}\, dx\, dy \tag{1}$$

stationary. Here $(g_{ik})_{ik}$ is a positive definite and $(b_{ik})_{ik}$ a skew-symmetric matrix. The Euler-Lagrange equations of I are given by

$$\Delta u^i + \Gamma^i_{kl}\nabla u^k \nabla u^l = g^{im}(b_{mk,l} + b_{kl,m} + b_{lm,k})\det(\nabla u^k, \nabla u^l) \tag{2}$$

with $(g^{ik})_{ik} = (g_{ik})^{-1}_{ik}$, $g_{kl,m} = \partial g_{kl}/\partial u_m$, and the Christoffel symbols Γ^i_{kl}. One can interpret (2) as the equation for a surface of prescribed mean curvature in a Riemannian manifold. The two-dimensional conformally invariant variational problems especially give rise to conformal maps between surfaces, parametric minimal surfaces in Riemannian manifolds, and harmonic maps from a surface into a Riemannian manifold.

For surfaces of prescribed mean curvature in $\mathbb{R}^3$ one considers the following functional of Heinz and Hildebrandt

$$I(u) = \frac{1}{2}\int_\Omega \{|\nabla u|^2 + Q(u)\cdot u_x \wedge u_y\}\, dx\, dy \tag{3}$$

for functions $u\colon \Omega \to \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^2$, where the vector field Q satisfies $\operatorname{div} Q(u) = 4H(u)$ with the mean curvature H and $\wedge$ denotes the cross product.

The functional (3) has Rellich's *H-surface system*

$$\Delta u = 2H(u) u_x \wedge u_y \quad \text{in } \Omega \tag{4}$$

as its Euler-Lagrange equation.

For harmonic maps one considers the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega g_{ik}(u_x^i u_x^k + u_y^i u_y^k)\,dx\,dy \tag{5}$$

generating the *harmonic functions*

$$\Delta u^i + \Gamma^i_{kl}(u_x^k u_x^l + u_y^k u_y^l) = 0, \qquad i = 1, \ldots, d. \tag{6}$$

The author discusses holomorphic quadratic functionals and gives a new proof of H. Hopf's celebrated theorem that a closed surface of genus 0 that is immersed with constant mean curvature into $\mathbb{R}^3$ is necessarily a sphere.

Using the Darboux system of equations

$$\begin{aligned}
&\Delta u + \left(\Gamma^1_{11} + \frac{1}{2}\frac{\partial}{\partial u}\log K\right)|\nabla u|^2 \\
&\qquad + \left(2\Gamma^1_{12} + \frac{1}{2}\frac{\partial}{\partial v}\log K\right)(u_x v_x + u_y v_y) + \Gamma^1_{22}|\nabla v|^2 = 0, \\
&\Delta v + \Gamma^2_{11}|\nabla u|^2 + \left(2\Gamma^2_{12} + \frac{1}{2}\frac{\partial}{\partial u}\log K\right)(u_x v_x + u_y v_y) \\
&\qquad + \left(\Gamma^2_{22} + \frac{1}{2}\frac{\partial}{\partial v}\log K\right)|\nabla v|^2 = 0,
\end{aligned} \tag{7}$$

a beautiful proof of Liebmann's theorem is given.

The second chapter is devoted to regularity questions. It begins with the introduction of harmonic coordinates due to Karcher and Jost and continues with the fundamental uniqueness theorem of Jäger and Kaul for harmonic maps. Then Grüter's ingenious method is utilized to show continuity of weak solutions via the monotonicity formula: At first continuity of weak minimal surfaces (in the interior and up to the free boundary) is shown. Then weak solutions of the H-surface system (4) given in weakly conformal parameters

$$u_x \cdot u_y = 0, \quad |u_x| = |u_y| \qquad \text{a.e. in } \Omega \tag{8}$$

are shown to be continuous. Finally, a general regularity result for two-dimensional geometric variational problems is presented. The celebrated theorem of Sacks and Uhlenbeck on the removability of isolated singularities is also derived.

Higher regularity, together with $C^{2,\alpha}$-estimates for harmonic maps, are then studied. Centrally important is the theorem of Hildebrandt, Kaul, and Widman: "Let N be a complete Riemannian manifold with a complete C^2-submanifold M. If $u \in H^{1,2}(D, N)$ is a continuous, weakly harmonic map with free boundary M, then $u \in C^{1,\alpha}(\overline{D}, N)$ holds true."

With the aid of the fundamental method of Heinz, the author derives gradient bounds for harmonic maps up to the free boundary. Also, the Hartman-Wintner lemma for asymptotic expansions at singular points (in the interior and on the boundary) of the form

$$u_z = \tfrac{1}{2}(u_x - iu_y) = a(z - z_0)^m + \circ(|z - z_0|^m), \qquad z \to z_0, \tag{9}$$

is explained.

The last section of Chapter 2 is devoted to estimates from below for the Jacobian of univalent harmonic mappings. The interior estimates are valid up to the boundary under convexity assumptions. Heinz discovered the fundamental importance of these kind of estimates for curvature bounds and $C^{2,\alpha}$-estimates for Monge-Ampère equations in connection with the system (7) and provided basic analytic tools to estimate the Jacobian from below.

Chapter 3 deals with conformal representation of surfaces homeomorphic to the sphere S^2, circular domains, and closed surfaces of higher genus. The proof is given by direct variational methods and not as usual by uniformization, completing a fragmentary proof of Morrey. The continuous and differentiable boundary correspondence is also directly attained.

Chapter 4 is devoted to existence results. Having established a local existence theorem, a harmonic map representing a saddle point of a certain functional is constructed, and many corollaries are presented. For this theorem originally due to Sacks and Uhlenbeck, the author invented a new interesting proof imitating the curve shortening process for the construction of unstable closed geodesics. Additionally, boundary conditions are imposed, especially of Plateau type. The author provides a new proof of the celebrated moutain pass lemma for minimal surfaces in Riemannian manifolds originally due to Ströhmer: "If u_1 and u_2 are two strict relative minima (w.r.t. the C^0- or $H^{1,2}$-topology), then there exists a third minimal surface u_3 which is unstable." In this chapter also, the Plateau-Douglas problem in Riemannian manifolds is beautifully treated, namely, to span a minimal surface of higher topological type into a system of Jordan curves.

In the beginning of Chapter 5 a local existence result for disc-type harmonic diffeomorphisms between Riemannian manifolds is proved by a continuity method, which was invented by the author. This yields the basis for the following global result of Jost and Schoen: "Let Σ_1 and Σ_2 be compact surfaces without boundary and $h\colon \Sigma_1 \to \Sigma_2$ a diffeomorphism. Then there exists a harmonic diffeomorphism $u\colon \Sigma_1 \to \Sigma_2$ isotopic to h, which is energy-minimizing in this class."

Also, a global harmonic diffeomorphism with Dirichlet boundary data is constructed, and the following interesting result of Schoen, Yau, and Sampson is proved: "Let $u\colon \Sigma_1 \to \Sigma_2$ be a harmonic map between closed oriented surfaces of the same genus with $\deg(u) = \pm 1$, and let the curvature K_2 of Σ_2 be nonpositive. Then u is a diffeomorphism."

With the aid of this well-developed theory of harmonic maps, the author provides a new approach to the Teichmüller theory in Chapter 6. Controlling the corresponding holomorphic quadratic differential, the basic structures of the Teichmüller space differential, the basic structures of the Teichmüller space are obtained, namely, the topological, differentiable, complex, metric, and Kählerian. Harmonic maps are better suited in this context than quasi-conformal maps, since the first class of functions can be better analytically controlled.

This excellent monograph, which is deeply rooted in the mathematical areas of calculus of variations, nonlinear partial differential equations, differential geometry, complex analysis, and topology, covers a broad, central region of mathematics in great depth. In this treatise the author has reunited mathematical areas that had been a unit in the times of Riemann, and he has invested the mathematical strength now required. The book is well written and highly recommended to every mathematician with interests in one of the areas mentioned above.

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