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BOOK REVIEW

Additive number theory of polynomials over a finite field, by Gove W. Effinger and David R. Hayes. Clarendon Press, Oxford, 157 pp., $45.00. ISBN 0-19-853583-x

There cannot be many mathematicians who have not heard of the two most celebrated areas of additive number theory, the Goldbach (1742) conjectures and Waring's (1770) problem. The modern intepretation of the Goldbach conjectures is that every even number is the sum of at most two primes and that every odd number greater than one is the sum of at most three primes.

Waring's problem in its original form is the determination, for each $n > 1$, of $g(n)$, the smallest number s such that every number is the sum of at most s positive nth powers. It turns out that a few peculiar small numbers such as

$$2^n[(3/2)^n] - 1$$

need a relatively large number of nth powers to represent them, so the value of $g(n)$ has been established for most values of n. In view of this, the modern form of Waring's problem has become the determination of $G(n)$, the smallest s such that every sufficiently large number can be written as the sum of at most s positive nth powers.

Over the last seventy years there has been a considerable amount of work on these problems, much of it through an analytic technique introduced in a series of seminal papers by Hardy and Littlewood [3-10] and then developed and refined by a number of researchers, most notably Vinogradov [13] and Davenport [2]. The Hardy-Littlewood method has many applications to other problems, many generalizations, and variants, for example, to algebraic number fields, and many of the associated technical devices have relevance to other areas of analytic number theory, such as the theory of the Riemann zeta function and the theory of diophantine approximation. There have been a number of accounts of the method over the years in its various forms and applications, e.g., by Landau [11], Vinogradov [13], Davenport [1], and Vaughan [12]. The basic principle of the method is that one can use the orthogonality of the additive characters $e(2\pi i n *)$ on $\mathbb{R}/\mathbb{Z}$ to count solutions of an equation via appropriate generating functions. These functions have peaks at rational points with relatively small denominators and can be shown to be of smaller order at points that do not have a good fractional approximation with such a denominator. Under suitable conditions this leads to an asymptotic formula for the number of representations of a number in the required form. One way of formally and abstractly describing the relationship between $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$, or, rather, the unit circle $\mathbb{T}$, is to say that $\mathbb{T}$ is the Pontryagin dual of $\mathbb{Z}$.

This book is concerned with versions of these questions formulated not for the ring of integers $\mathbb{Z}$ but for the ring $\mathbf{A} = \mathbb{F}_q(T)$ of polynomials over the finite field of q elements. That $\mathbf{A}$ should be considered for these particular properties is largely a consequence of being able to mirror many of the fundamental properties of $\mathbb{Z}$; in particular, there are natural analogues of the additive characters on $\mathbb{R}/\mathbb{Z}$. The authors follow the adèlic route pioneered by Weil [14, 15]. They take k to be the field of fractions of $\mathbf{A}$ and $\mathbb{A}_k$ to be the adèle ring over k. Then they consider the adèle class group $\mathbb{A}_k/k$ of k. The point is that, considered as a discrete additive group, this is the Pontryagin dual of k. Moreover, the corresponding generating functions have peaks at "rational points" with "small" denominators and can be shown to be smaller elsewhere. Much of the analysis is analogous to, and even deliberately imitative of, that in the classical cases.

Naturally this approach involves considerable abstraction. In one respect, at least, this is rather unfortunate, since the basic problems considered can be formulated in an elementary fashion and questions such as whether a given polynomial can be written as the sum of three irreducible polynomials, say, have a broad appeal. Moreover, the authors state in the preface that their exposition is directed towards a first or second year postgraduate student who knows basic number theory and has mastered a standard abstract algebra course. Well, in my experience, there are very few such students who will have mastered Haar measure on locally compact groups, Hensel's lemma, or the concepts of discrete valuations of a field and its associated places. Yet these are all topics that the unprepared reader will have to seek help on elsewhere.

In spite of the title of the book it deals only with the two questions mentioned above, much of the space being taken up with developing such things as global and local Radon-Nikodym derivatives, the idèle group of k, and L-functions over k. Moreover, in the analogue of Waring's problem, only the simplest upper bounds are established for the analogue of $G(n)$, namely, those that correspond to Weyl's inequality and Hua's lemma in the classical problem. There is no hint as to whether there is any possibility of improving the upper bound for the required number of summands, either by imitating some of the more sophisticated techniques used in the classical situation or by introducing new ideas directly. It would be useful also to have included some indication of what one might expect to be best possible. Of course, there are extra parameters to be taken into consideration compared with the classical case, such as the degrees of the polynomials and the order of the ground field, but a close study of the singular series and integral, i.e., of solubility at each place, ought to give quite a good indication of what might be best possible.

There are a number of curiosities. On pages 15 and 16 there is a strange discussion on the use of the word singular in "singular series". The authors seem to overlook that Weierstrass undoubtedly introduced the term singular for the singular points of an analytic function precisely because the points are most singular.

On page 17 it is asserted that Hardy and Littlewood were certainly aware that their singular series were products of p-adic density functions. As far as I know, Hardy and Littlewood never used the term p-adic and were probably not aware that the individual terms in the Euler product for their singular series could be interpreted as the density of solutions in $\mathbb{Q}_p$ of the underlying equation. By the time such an interpretation became fashionable in the 1960s, Hardy had been dead for well over a decade and Littlewood had turned long ago to other things.

For anyone wishing to absorb some of the more algebraic developments in number theory and willing to make the further investment in mastering the assumed material, this book would be an excellent and stimulating way in. For someone who is already an expert, it is probably less attractive. The questions considered are unlikely to be of central interest, and the underlying techniques described, unlike the classical situation where many of the techniques have wide application to the theory of the Riemann zeta function, to diophantine approximation, and to other problems in additive number theory, almost certainly will not throw much light on fundamental problems in algebraic number theory or algebraic geometry. Those nonexperts not willing to invest the time but hoping to gain some insight into the questions studied in the book will find the high level of abstraction a frustating barrier.

The style is somewhat terse but is quite clear, and there are stimulating exercises at the end of each section.

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