BOOK REVIEWS

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Elements of KK-theory, by Kjeld Knudsen Jensen and Klaus Thomsen. Mathematics: Theory & Applications, Birkhäuser, Boston, Basel, and Berlin, 1991, viii + 202 pp., $49.50. ISBN 0-8176-3496-7, ISBN 3-7643-3496-7

What is KK-theory and why is it useful? The answers to these questions are, in brief, that KK is a bivariant version of topological K-theory and that it provides a useful framework for the study of the index theory of elliptic pseudodifferential operators, in its most general and refined formulations. In turn, index theory provides a means for "counting" the number of solutions of a linear elliptic partial differential equation on a manifold in terms of topological data. Much to my surprise, I found that neither these basic questions nor some sort of answers to either of them appear anywhere in this book. Therefore, to explain what this book is about, I will try in my own way first to explain the purpose of KK-theory and then to describe the niche that this book attempts to fill.

The subject of topological K-theory was invented by Atiyah and Hirzebruch [AH], who copied constructions of Grothendieck in algebraic geometry. Atiyah and Hirzebruch defined a functor K^0 from compact (Hausdorff) topological spaces to abelian groups by letting $K^0(X)$ be the group of formal differences of isomorphism classes of vector bundles over a space X, with group operation defined by the obvious formula

$$([E^0]-[E^1])+([F^0]-[F^1])=[E^0\oplus F^0]-[E^1\oplus F^1].$$

Actually, one gets two functors this way, KO^0 and KU^0, depending on whether one uses real or complex vector bundles, but the essential observation of Atiyah and Hirzebruch was that both extend to "generalized cohomology theories" on the category of compact topological spaces, with the remarkable property (known as Bott Periodicity) that $KO^j(X)$ and $KU^j(X)$ are periodic in j, with period 8 and 2, respectively. They already realized that the subject of K-theory had something to do with the index theory of elliptic operators. This connection was made much more explicit through the work of Atiyah and Singer [AS], whose first published proof [AS, I] of their celebrated Index Theorem was based on Gysin (or "wrong-way") maps in K-theory—in other words, on maps $f_!\colon K^j(X)\to K^j(Y)$ induced by a map of manifolds $f\colon X\to Y$. Atiyah and Singer were also forced to consider the case of noncompact manifolds (in order to treat a manifold and its cotangent bundle on an equal footing), so they worked with *K-theory with compact supports* for locally compact spaces X, in which $K^0(X)$ is a group of equivalence classes of diagrams

$$\begin{array}{ccc} E^0 & \xrightarrow{f} & E^1 \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

where E^0 and E^1 are vector bundles over X and f is a morphism of vector bundles, which is an isomorphism off a compact set. Such a diagram represents 0 in $K^0(X)$ if f is an isomorphism, and two such diagrams are equivalent if there is a homotopy (through such diagrams) between them. The group operation is induced by the direct sum operation. When X is compact, such a diagram $E^0 \xrightarrow{f} E^1$ corresponds simply to $[E^0]-[E^1]$ in the Atiyah-Hirzebruch definition, since one can deform f continuously to 0.

As any algebraic topologist knows, a Gysin map $f_! : H^j(X) \to H^j(Y)$ induced by an orientation-preserving map of manifolds $f : X^n \to Y^n$ is really a composite

$$H^j(X) \xrightarrow{\text{Poincaré duality}} H_{n-j}(X) \xrightarrow{f_*} H_{n-j}(Y) \xrightarrow{\text{Poincaré duality}} H^j(Y).$$

Indeed, in Grothendieck's version of K-theory for algebraic varieties, the wrong-way maps of Atiyah and Singer corresponded to "right-way" maps on *dual* K-groups with the opposite functoriality. Hence, the work of Atiyah and Singer suggested that a still more elegant approach to index theory should be based not on the existing K-theory, which was a cohomology theory, but rather on the dual theory, *K-homology*. Indeed, Atiyah made a celebrated suggestion of trying to construct an explicit realization of this homology theory by defining groups $\mathrm{Ell}_*(X)$ of equivalence classes of suitable abstractions of elliptic operators over the space X.

The problem of giving a rigorous definition of what one means by "equivalence classes of suitable abstractions of elliptic operators over a space X" was solved by Kasparov [K1] and by Brown, Douglas, and Fillmore [BDF], working independently. The papers [K1] and [BDF] can be said to mark the beginnings of KK-theory.

At first, the work of Kasparov and the work of Brown-Douglas-Fillmore seemed to be quite different. The one feature they had in common, which turned out to be essential, was their reliance on the theory of C*-algebras. A *C*-algebra* is a Banach algebra A with an involution $*$ (an antilinear antiautomorphism), isometrically isomorphic to an algebra of bounded Hilbert-space operators closed under the operator norm and the adjoint operation $*$. (Isomorphisms in the category of $*$-algebras are required to be compatible with the involution.) By the foundational work of Gel′fand and Naimark, the functor $X \mapsto C_0(X)$ gives a *contravariant* equivalence of categories from the category of locally compact Hausdorff spaces and proper continuous maps to the category of commutative (complex) C*-algebras. Furthermore, the Serre-Swan Theorem identifies $K^{-j}(X)$, the K-theory of X with compact supports, with $K_j(C_0(X))$, the topological K-theory in the sense of Karoubi (now a *covariant* theory on algebras) of the C*-algebra $C_0(X)$. (This works equally well over the reals and the complexes.) So *K-homology* of locally compact spaces X should come from a *contravariant* K-functor on C*-algebras, as applied to the commutative C*-algebra $C_0(X)$.

Kasparov's approach to the construction of analytic K-homology meshed quite closely with the Atiyah-Singer approach to index theory. The group $K_0(X)$, which one can also write as $K^0(A)$, where $A = C_0(X)$, was constructed from equivalence classes of triples $(\mathscr{H}, \varphi, F)$, where $\mathscr{H} = \mathscr{H}^0 \oplus \mathscr{H}^1$ is a $\mathbb{Z}/2$-graded Hilbert space, $\varphi : A \to \mathscr{L}(\mathscr{H})$ is a $*$-homomorphism of degree 0 (i.e., $\varphi(a)$ preserves the grading of $\mathscr{H}$ for $a \in A$), and $F \in \mathscr{L}(\mathscr{H})$ is an operator of degree 1 (i.e., reversing the grading of $\mathscr{H}$), which is *almost* a selfadjoint unitary commuting with A. In other words, one requires

$$(*) \qquad \left.\begin{array}{l} \varphi(a)(F^* - F) \\ \varphi(a)(F^2 - 1) \\ [\varphi(a), F] \end{array}\right\} \in \mathscr{K}, \qquad a \in A.$$

Here $\mathscr{K} = \mathscr{K}(\mathscr{H})$ is the algebra of compact operators. Two such triples are to be viewed as equivalent if there is a homotopy from one to the other. The group operation comes from direct sum. The conditions $(*)$ are indeed satisfied when $A = C_0(X)$ for some manifold X, $\mathscr{H}^j = L^2(E^j)$, the L^2-sections of a vector bundle E^j over X, φ is pointwise multiplication of L^2-sections by continuous functions vanishing at infinity, and $F = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right)$, where D is an elliptic pseudodifferential operator of order 0 from sections of E^0 to sections of E^1. (If X is noncompact, one needs a mild condition on D to obtain boundedness, but this can always be achieved by a homotopy that does not change the index.) So the elliptic operator D defines a class $[D] \in K_0(X)$. (A technical point: In the cases of greatest geometric interest, one is interested not in an operator of order 0 but in a differential operator D' of order 1, for instance, the Dirac operator, the signature operator, or the $\bar{\partial}$ operator. However, one can always replace such an operator D' by the operator $D = D'(1 + D'^2)^{-1/2}$, which has the same index. The new operator is now of order 0 but is usually no longer a differential operator, only a pseudodifferential operator. For most purposes this makes no difference in the theory.) When X is compact, D is a Fredholm operator, and computing $c_*([D]) \in K_0(pt) = \mathbb{Z}$, where $c\colon X \to pt$ collapses X to a point, corresponds to taking the index of D, which is exactly the number computed by the Atiyah-Singer Theorem.

Brown, Douglas, and Fillmore, on the other hand, took as their basic object $K_1(X)$, which they realized as $\operatorname{Ext}(X)$, a group of equivalence classes of extensions of C*-algebras

$$0 \to \mathscr{K} \to E \to A \to 0,$$

where again $A = C_0(X)$. The model for this situation in index theory comes from taking E to be the (C*-algebra closure of the) algebra of pseudodifferential operators of order 0 on some compact manifold M, X to be the unit sphere bundle of the cotangent bundle T^*M, and $E \to A$ to be the map sending an operator to its principal symbol.

The great advance that unified the two approaches came in Kasparov's remarkable paper [K2], which defined for the first time the bivariant groups $KK(A, B)$. The beauty of Kasparov's definition was that with only a small modification in $(*)$, one could encompass both K-homology and K-cohomology in a single theory. Furthermore, KK-theory is ideally suited for studying fancier versions of index theory. To be more specific, let A and B be two (separable) C*-algebras. These can be algebras over either the reals or complexes, and both can be taken to be $\mathbb{Z}/2$-graded, though the case where the gradings are trivial is already of considerable interest. Then $KK(A, B)$ is again the group (under direct sum) of homotopy classes of triples $(E = E^0 \oplus E^1, \varphi, F)$ satisfying $(*)$ as before, the difference being that E is what is called a *Hilbert B-module* and that $\mathscr{L}(E)$, $\mathscr{K}(E)$ are interpreted in the sense of the theory of such modules, and the commutator in $(*)$ has to be interpreted in the graded sense. In particular, φ and F commute with the action of B, and the inner product on E takes its values in E instead of in $\mathbb{R}$ or $\mathbb{C}$. When B is just the scalars with the trivial grading, a Hilbert B-module is just a Hilbert space, and one recovers Kasparov's old definition of $K^0(A)$. But if A consists of the scalars and the map φ is just scalar multiplication, such a triple is defined by a $\mathbb{Z}/2$-graded Hilbert B-module E and an operator $F \in \mathscr{L}(\mathscr{E})$ of degree 1. The conditions $(*)$ reduce to assuming that $F^* - F$ and $F^2 - 1$ are B-compact. It turns out that one can always make a homotopy to the case where $F = F^*$, i.e., $F = \left(\begin{smallmatrix} 0 & f^* \\ f & 0 \end{smallmatrix}\right)$ for some $f : E^0 \to E^1$. When $B = C_0(X)$, a special case of such a triple is obtained by taking E^0 and E^1 to be (the sections of) vector bundles over X and f to be a morphism of vector bundles; the condition that $F^2 - 1$ be compact with respect to B then translates into the Atiyah-Singer condition that f be an isomorphism off a compact set. Furthermore, one can show that any triple is equivalent to one of this form, so $KK(A, B)$ is just the K-theory of X with compact supports in this case.

If it were just for the fact that the definition of KK unifies the definitions of K^0 and K_0, KK would have just remained a curiosity. But in fact Kasparov's paper went much further. For one thing, Kasparov showed that the various groups K^j and K_j can also all be obtained from the same construction, merely by replacing A or B by its (graded) tensor product with an appropriate Clifford algebra. Bott periodicity then becomes a manifestation of an algebraic periodicity of the Clifford algebras, as had already been pointed out in the case of the groups $KO^*(X)$ by Atiyah, Bott, and Shapiro in the fundamental paper [ABS]. Secondly, Kasparov showed that one could generalize the construction of the Brown-Douglas-Fillmore Ext-groups and define a monoid $\mathrm{Ext}(A, B)$ out of equivalence classes of C*-algebra extensions

$$0 \to \mathscr{K} \otimes B \to E \to A \to 0,$$

when one divides out by the split extensions. This monoid is a group if A is nuclear (abelian and type I C*-algebras are nuclear; the C*-algebra of a discrete group is nuclear if and only if the group is amenable), but not in general. However, Kasparov showed that the group $\mathrm{Ext}^{-1}(A, B)$ of invertible elements in $\mathrm{Ext}(A, B)$ is naturally isomorphic to $KK^1(A, B)$, so that there is a natural equivalence between the two approaches to analytically defined K-homology. Finally—and this was the most important part of Kasparov's paper, though it is the hardest part for the beginner to appreciate—Kasparov constructed a bilinear and associative product

$$KK(A, B) \times KK(B, C) \to KK(A, C),$$

which can be used to reconstruct all the classical products of topological K-theory (the cup product on cohomology, the cap product making homology a module over cohomology, and the slant product). Kasparov's construction of the product depended on one hard technical result (the "Kasparov Technical Theorem") and amounted to a generalization of the Atiyah-Singer "#-product" for elliptic operators.

What then is the importance of KK-theory? The real utility of Kasparov's construction comes from the fact that certain generalizations of elliptic operators naturally give classes in $KK(A, B)$ for suitable C*-algebras A and B, which in turn can be used via the Kasparov product to obtain maps of K-groups. The simplest example comes from the index theory for families of elliptic operators [AS, IV]. If M is a compact manifold and X some other compact parameter space, a family of elliptic operators $\{D_x\}_{x\in X}$ over M parameterized by X naturally defines a class $[D]$ in $KK(C(M), C(X))$. The index of this family in the sense of Atiyah and Singer is a class in $K^0(X)$, which roughly speaking is defined by taking the formal difference of the vector bundles $\{\ker D_x\}$ and $\{\ker D_x^*\}$ over X. (To be technically correct, one first has to modify the family of operators without changing the K-theory class of its symbol so that they give genuine bundles.) From the Kasparov point of view, the index is again obtained by taking $c_*([D]) \in KK(C(pt), C(X)) = K^0(X)$, where $c: M \to pt$ is the collapse map. A fancier example comes from replacing $C(X)$ by a noncommutative C*-algebra B, for example, the C*-algebra of a discrete group, and replacing a family of elliptic operators by a B-linear elliptic operator in the sense of Miščenko and Fomenko [MF]. This yields a class in $KK(C(M), B)$ and an index in $KK(C(pt), B) = K_0(B)$. Still fancier examples come from index theory on foliated manifolds or manifolds with group actions, where now $C(M)$ is replaced by a noncommutative C*-algebra reflecting the structure of the foliation or group action. Finally, Gysin maps, as in the original Atiyah-Singer approach to the Index Theorem, can be viewed as KK-classes; to a K-oriented map $f\colon X \to Y$ one can attach a class $f_! \in KK(C_0(X), C_0(Y))$ [CS, §2] such that the Kasparov product with this class is the Gysin map used by Atiyah and Singer. Again, this can be done in a noncommutative context with foliated manifolds [HS]. While there is no point here in trying to give a catalogue of all the cases where KK-theory has proved useful, we should at least mention that Kasparov's important work [K3] on the Novikov Conjecture, which deals with the classification of nonsimply connected manifolds, would have been impossible without KK-index theory.

What about the book? The Jensen-Thomsen monograph gives admirably clear presentations of the theory of Hilbert modules, of the Kasparov Technical Theorem, and of the construction of the Kasparov product. In addition, it does a fine job with some newer approaches, due to Cuntz, Higson, Skandalis, Thomsen, and Zekri, to the construction and basic properties of the KK-groups, and the exercises should help the reader to grasp some of the main points. The book, however, never once mentions the index theory of elliptic operators, and the names of Atiyah, Bott, Karoubi, and Singer do not seem to occur anywhere in the text or in the subject index. Thus, the reader who picks up this book hoping to find out something about the main *motivation* for KK-theory will know no more after reading it than he did before. Nevertheless, this book is a useful addition to the literature. For the beginner who wants to learn about KK, however, this is the *second* book to read. Blackadar's exposition [B], nicely reviewed in this journal by Lance [L], is the correct place to begin. Someone who can read French should also probably first read the concise but well-written survey by Fack [F]. Jensen and Thomsen have wisely chosen not to try to duplicate Blackadar's book, to the point where they have tried not even to overlap with Blackadar's bibliography. Once one understands the purpose and motivation of KK, Jensen-Thomsen is a good source for reading about some of the technical details. It is far from being exhaustive; the real and equivariant cases of the theory, for instance, are never even mentioned. But for the topics covered, particularly the coincidence of several alternative definitions of the KK-groups, and for the existence and associativity of the Kasparov product from several points of view, the authors give a readable and accurate exposition.

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