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Pluripotential theory, by Maciej Klimek. London Math. Soc. Monographs (N.S.), vol. 6, Oxford Univ. Press, New York, 1991, xiv+266 pp., $59.95. ISBN 0-19-853568-6

We will first discuss *potential* theory in one complex variable and then the higher-dimensional *pluripotential* theory.

At first, recall the concept of a harmonic function.

Holomorphic functions in the complex plane satisfy the Cauchy Riemann equation $\partial f/\partial \overline{z} = 0$. One sees immediately that the two-dimensional Laplacian

$$\partial^2/\partial x^2 + \partial^2/\partial y^2 =: \Delta = 4\partial/\partial z\partial/\partial \overline{z}\,.$$

Hence holomorphic functions are also solutions of the homogeneous Laplace equation. In fact both the real and imaginary parts of a holomorphic function are solutions to the Laplace equation. In general, in any dimension, the solutions of the homogeneous Laplace equation are called harmonic functions. A special property in one complex dimension is that the real-valued harmonic functions on the complex plane are the real parts of holomorphic functions.

In $\mathbb{R}^3$ the solutions of the homogeneous Laplace equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$$

are called harmonic functions or *Newtonian potentials*. Their gradient is a vectorfield called *force*, and specific examples are given by *gravitational potential* and *gravitational force* or *electric potential* and *electric force*. The basic example of a Newtonian potential in $\mathbb{R}^3$ is $1/\|x - x_0\|$ corresponding to a point mass or point charge at x_0. Then the force, given by the gradient of the potential, decays like one over square of the distance.

The same terminology is carried over to $\mathbb{C} = \mathbb{R}^2$.

In $\mathbb{R}^2$ the solutions of the homogeneous Laplace equation are called *logarithmic potentials.* Now the basic example is $2\pi \log\|x - x_0\|$ corresponding to a "*unit point mass*" at x_0. In this case the "*force*", given by the gradient of the potential, decays like one over distance.

The function $\log\|x - x_0\|$ is harmonic when $x \neq x_0$ and has a singularity at x_0. Including x_0 in the domain of the function, $\log\|x - x_0\|$ is said rather to be *subharmonic* for all x. Similarly one calls the extension of $-\log\|x - x_0\|$ to the whole complex plane *superharmonic.* A similar remark can be made in $\mathbb{R}^3$, in fact, just as well in $\mathbb{R}^n$, $n \geq 3$, but this is not our concern here.

Generalizing this, one is naturally lead to subharmonic functions from harmonic functions when one considers masses spread over a region rather than point masses.

Subharmonic functions were introduced by F. Riesz in 1922. A common way to define a subharmonic function in general is by requiring that whenever a harmonic function is larger than a subharmonic function on the boundary of a region, the harmonic function is larger than the subharmonic function on the interior as well. In addition, one requires the technical hypothesis that the function be upper semicontinuous and can take values in $\mathbb{R} \cup \{-\infty\}$ but not identically $-\infty$. The basic properties of subharmonic functions are discussed clearly in Chapter 2 of Klimek's book.

The Fundamental Theorem of F. Riesz is that a subharmonic function f locally is the convolution of its Laplacian μ with the logarithmic potential of zero, $\log\|x\|$ (Fundamental solution), up to a harmonic addition h,

$$f(x) = \iint f(x - y) * \log\|y\| \, d\mu(y) + h(x).$$

Hence it is not unreasonable to say that a subharmonic function is a potential of its Laplacian.

Subharmonic functions were introduced in order to solve the Dirichlet problem in a rigorous way: Find a harmonic function in a region with given values at the boundary. This is carried out in Chapter 2.

Notice that harmonic functions have a certain special position among all subharmonic functions. We say that a subharmonic function is *maximal* if whenever a subharmonic function lies under it on the boundary of a region this subharmonic function stays under in the interior as well. The maximal subharmonic functions are also precisely those subharmonic functions ϕ for which $\Delta\phi = 0$.

This point of view is useful for solving the Dirichlet problem using the Perron method and is the point of view adopted in Klimek's book: To find a function with given boundary values which satisfies the Laplace equation in the interior, take the maximum of the subharmonic functions with boundary value below the given one. Hence, the solution to the Dirichlet problem is the maximal subharmonic function with the given boundary values.

I will next turn to the higher-dimensional complex analysis. This is closely related to the one variable theory—one just considers holomorphic functions of more than one complex variable and requires that they are holomorphic in each variable depending holomorphically on additional parameters.

There are some new phenomena that sets the theory of higher-dimensional complex analysis apart from the theory of one complex variable. The main cause of this is the remarkable discovery by Hartogs (1874–1943) in 1905 that holomorphic functions of more than one variable cannot have isolated singularities, such as $1/z$, in one variable. This follows from the Cauchy Integral formula after a moment's thought: To remove a singularity at $(0, 0)$ use the formula

$$f(z, w) = \int_{|\eta|=1} \frac{f(\eta, w)}{z-\eta}\, d\eta .$$

The second chapter of the book also deals with this phenomenon. An even more remarkable version of this phenomenon is that holomorphic functions extend to the outside of certain nonconvex regions. The basic examples are H-shaped Hartogs figures in $\mathbb{C}^2$. These are of the form

$$H = \{|z| < 1+\varepsilon ,\ |w| < \varepsilon\} \cup \{1+\varepsilon > |z| > 1-\varepsilon ,\ |w| < 1\} .$$

This time the holomorphic functions on H extend to the convex hull using the exact same integral as before.

Hartogs analyzed this phenomenon. He considered a slight generalization of this basic Hartogs figure, namely, the Hartogs domains

$$H = \{(z, w) \in \mathbb{C}^2;\ z \in U,\ |w| < e^{-\phi(z)}\} .$$

He showed that holomorphic functions on these generalized H's extend beyond the boundary precisely when the function ϕ fails to be a *Hartogs function.* Loosely speaking Hartogs functions are the lattice generated by the functions $c\log|f|$, where $c > 0$ and f is holomorphic. A Hartogs function is the same as a subharmonic function—but this was about 15 years before subharmonic functions were invented—and, anyway, this equivalence was not proved until 1956 by Bremermann.

This brought the topic of subharmonic functions into the theory of several complex variables.

These discoveries showed that in higher dimension not all domains are natural domains of definition for holomorphic functions. This is because for some domains all holomorphic functions extend to a given strictly larger domain. It is customary to call all domains that are natural domains of definition for holomorphic functions domains of holomorphy. Hartogs's discovery then showed that Hartogs domains in $\mathbb{C}^2$ are domains of holomorphy precisely when ϕ is a Hartogs function/subharmonic.

Notice that ϕ can be interpreted as $-\log(\delta)$ on the base $U = U * 0$, where δ denotes the distance from a point to the boundary in the w-direction.

It took the genius of Oka to see how this could be used for general domains. The Hartogs domains are, after all, rather special.

He proved in 1942 that a general domain U in $\mathbb{C}^2$ is a domain of holomorphy if and only if the function $-\log(\text{dist})$ is subharmonic on each complex line. Here dist denotes the Euclidean distance to the boundary of U. Oka called such functions *pseudoconvex.* This class of functions was simultaneously introduced by Lelong, who called them *plurisubharmonic.* So to be more precise, an upper semicontinuous function whose restriction to any complex line is subharmonic is called plurisubharmonic. The terminology of Lelong has prevailed, so these functions are nowadays called plurisubharmonic.

Lelong's motivation to introduce the plurisubharmonic functions is rooted in a similar way in the Hartogs extension phenomenon. Levi (1910) observed that a smoothly bounded strongly convex domain in $\mathbb{C}^2$ is a domain of holomorphy. Such a domain has a defining function which is strongly convex. The property of being a domain of holomorphy is invariant under biholomorphic maps. The biholomorphic image of a strongly convex domain, therefore, is a domain of holomorphy. Carrying over the convex defining function gives us a plurisubharmonic defining function. Lelong's motivation for introducing plurisubharmonic functions was that they could be used to describe in this way a complex analytic version of convexity. However, instead of using these functions to study the Levi problem, Lelong's main use was based on the theorem that a function u is plurisubharmonic if and only if the current $dd^c u$ is positive and closed. (Here d^c is the complex conjugate version of d.) Currents are discussed in Chapter 3.

Actually, the main part of Oka's aforementioned work reduced to showing that a smoothly bounded domain which locally could be made strongly convex by a holomorphic change of coordinates was a domain of holomorphy. This work was very deep.

The first part of the book contains the first two chapters. In addition to the introductory material already mentioned, the second chapter also contains a careful discussion of plurisubharmonic functions.

Once plurisubharmonic functions were introduced by necessity, it was quite natural to ask whether they could be used in a way similar to subharmonic functions. So going back to the original reason for studying subharmonic functions, one can ask whether some sort of Dirichlet problem can be solved even for a simple domain like a ball. Indeed this was done by Bremermann in the 1950s. The problem is what happens, for instance, when one applies the Perron method in this case: Let f be a function on the boundary of a domain. Consider the supremum of all plurisubharmonic functions with values less than f on the boundary. One can call this supremum the solution of the Dirichlet problem. Bremermann showed that this is the *maximal* plurisubharmonic function with value f on the boundary: In analogy with one complex variable we say that a plurisubharmonic function is maximal if whenever a plurisubharmonic function lies under it on the boundary of a region this plurisubharmonic function stays under in the interior as well. As already indicated above in the discussion of the one variable case, Klimek's main objects are the *maximal* plurisubharmonic functions. These are introduced in Chapter 3.

Note that although one can think of the maximal plurisubharmonic functions as the closest possible to the Real Part of a holomorphic function, they might be far from that. For example, $\max\{\log(|z|^2+|w|^2)\}$ is maximal plurisubharmonic away from $(0, 0)$.

In analogy with one complex variable one can call a plurisubharmonic function ϕ a pluripotential, but one needs to decide of what it should be the pluripotential. For this one can seek a higher-dimensional version D of the Laplacian and then say that ϕ is the pluripotential of $D\phi$. Then one can develop a pluripotential theory.

It turns out that solutions to the Dirichlet problem above are under sufficient regularity conditions solutions of $(dd^c u)^n := dd^c u \wedge dd^c \wedge \cdots \wedge dd^c u = 0$. This is the main topic of Chapter 4. The rough idea is that if the function u is harmonic in at least one complex direction, it should be maximal and $(dd^c u)^n = 0$. Call this the n-dimensional Laplacian. If $Du := (dd^c u)^n = g$, u is called a pluripotential of g. This operator D is actually rather called the *complex Monge Ampere operator* (and not the Laplacian) because of its similarity with the real Monge Ampere operator. (While D is given by the determinant of the complex Hessian $\{\partial^2/\partial z_i \partial \overline{z}_j u\}$ of u, the real Monge Ampere operator is the determinant of the real Hessian.) The maximal plurisubharmonic functions are precisely those plurisubharmonic functions ϕ for which $(dd^c\phi)^n = 0$, i.e., the solutions of the Dirichlet problem using the so-called *Bremermann-Perron* method above.

As already indicated, the following result, treated in Chapter 4, by Bedford and Taylor (1976) is a central theorem in pluripotential theory:

Theorem 0.1. *Let Ω be an open subset of $\mathbb{C}^n$, and let u be a locally bounded plurisubharmonic function on Ω. Then u is maximal if and only if it satisfies the homogeneous Monge Ampere equation.*

Of other main results from Chapter 4 we mention a few, for example, the fact by Bedford and Taylor (1982) that (locally in L^∞) if we have a sequence of plurisubharmonic functions u_m converging monotonically up or down to a plurisubharmonic function u then $(dd^c u_m)^n \mapsto (dd^c u)^n$.

Bedford and Taylor also have proved (1982) the so-called Comparison Theorem for bounded plurisubharmonic functions: $\int_{u<v}(dd^c u)^n \leq \int_{u<v}(dd^c v)^n$ if $u > v$ on the boundary of the domain.

Lelong (1983) showed that every locally bounded plurisubharmonic function is the limit in L^1_{loc} of u_j with $(dd^c u_j)^n = 0$, thus making it clear that the above results could not easily be extended to unbounded functions.

In addition, let me mention Josefson's Theorem (1978) that if locally a set is contained in a pluripolar set, i.e., the set where a plurisubharmonic function equals $-\infty$, then there is a global plurisubharmonic function with this property. Bedford and Taylor reproved this result using the operator $(dd^c)^n$. Another of their results is that if a bounded function u is a limit of an increasing sequence of plurisubharmonic functions and u^* denotes the smallest upper semicontinuous function $\geq u$, then u^* differs from u only inside a pluripolar set.

As demonstrated by Lelong's example above, one central difficulty, and a task for the future, in this subject is how to define $(dd^c u)^n$ for u unbounded, at least for some mildly restricted class of functions.

Another direction to develop is to find good generalizations of F. Riesz's Theorem above; namely, one would like to approximate a plurisubharmonic function by a maximal plurisubharmonic function with only finitely many singular points. For one point, one has functions like $\log\|z\|$ and Lempert has another interesting such function on convex domains. For two or more points it is difficult to find such functions since the operator $(dd^c)^n$ is nonlinear. This topic is treated in Chapter 6.

In the last few years maximal plurisubharmonic functions have been used successfully in higher-dimensional complex dynamics; often these have logarithmic growth (a class of functions discussed in Chapter 5). This will undoubtedly continue.

The book by Klimek gives an excellent introduction to the subject of maximal plurisubharmonic functions. He develops the theory in detail from scratch, making the book suitable for use in a graduate course. The first two chapters have exercises at the end. It would have been useful to also have exercises after the remaining chapters.

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Analysis of Toeplitz operators, by A. Böttcher and B. Silbermann. Springer-Verlag, New York, 1990, 512 pp., $79.00. ISBN 3-540-52147-X

Given a complex-valued function ϕ on the real line $\mathbf{R}$, one can define the multiplication operator $M(\phi)$, acting on the Lebesgue space $L^2(\mathbf{R})$, by the recipe $M(\phi)\colon f \to \phi f$. To operator theorists these objects are useful and well understood, yet not without a certain charm due to a nice correspondence between properties of $M(\phi)$ and properties of ϕ. If you are a Fourier analyst, you will probably want to multiply "in the Fourier transform variable" as well, via the Fourier multiplier $D(\psi) = F^{-1}M(\psi)F$, where F is the Fourier transform on $L^2(\mathbf{R})$. Since F is unitary (thus preserving all Hilbert space structure), $D(\psi)$ and $M(\psi)$ are isomorphic as Hilbert space operators, but they relate very differently to the functional values of vectors in $L^2(\mathbf{R})$. On taking algebraic combinations of $M(\phi)$'s and $D(\psi)$'s, one has an interesting mix indeed, including the linear differential operators with variable coefficients: if $\psi(x) = x$, then $D(\psi) = -id/dx$. If you want to study bounded operators on $L^2(\mathbf{R})$, you must take ϕ and ψ in $L^\infty(\mathbf{R})$, but you are still left with bounded pseudodifferential operators, including several famous subclasses (all of which are generated by restricting either ϕ or ψ to be χ, the characteristic function of $[0, \infty)$): singular integral operators, Wiener-Hopf operators, Hankel operators, and, the subject of the book of Böttcher and Silbermann, the Toeplitz operators

$$D(\chi)M(\phi)D(\chi). \tag{1}$$

For most purposes, the preferred habitat of Toeplitz operators is the unit circle $\mathbf{T}$, where they look a bit more natural. Let us write L^2 for the Lebesgue space on $\mathbf{T}$ (with respect to normalized arc-length measure $d\theta/2\pi$), which comes equipped with the orthonormal basis $\{e^{in\theta}\colon n = 0, \pm 1, \pm 2, \dots\}$. The Hardy subspace H^2 is the closed linear span of the nonnegative frequencies $\{e^{in\theta}\colon n = 0, 1, 2, \dots\}$. The orthogonal projection P of L^2 onto H^2 is the analogue of $D(\chi)$ above, and the analogue of (1) is $PM(\phi)P$, where ϕ is now an L^∞ function on $\mathbf{T}$. Since the last operator annihilates the orthogonal