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BOOK REVIEW

Combinatorial homotopy and 4-dimensional complexes, by Hans Joachim Baues. Walter de Gruyter, Berlin, 1991, 379 pp., $89.50. ISBN 0-89925-697-x.

Before discussing the material in the book under review, let me try to describe its mathematical context, starting with the notions of topological space and homeomorphism. Some technical language will be used without definitions. At their first appearance, such terms will be placed in quotation marks, in the hope that the context together with the reader's background will convey the general idea.

Ideally, one wants a classification theorem, or at least a characterization of topological type for significantly occurring spaces, in terms of a short list of invariants. As a general goal, this is out of reach. A breakthrough idea appeared in the work of J. W. Alexander, in his study of the topological invariance of simplicial homology. He showed that spaces of the same homotopy type have isomorphic homologies. Let us review the definitions. Two continuous maps $f, g : X \to Y$ are *homotopic* if there is a continuous map $H : X \times [0, 1] \to Y$ extending f and g on the subspaces $X \times \{0\}$ and $X \times \{1\}$ respectively. Two spaces X, Y have the same *homotopy type* provided there are maps $f : X \to Y$ and $g : Y \to X$ which are inverses up to homotopy; gf is homotopic to the identity map of X, and likewise for fg.

Often, but not always, computable topological invariants are actually invariants of homotopy type. Furthermore, the possibility of a short list of homotopy invariants characterizing significantly occurring homotopy types seems more within reach than the same problem for topological types. Even so, there are still major difficulties.

Some understanding of the problems for characterizing general homotopy types can be gained when one realizes that the basic algebraic tools in homotopy theory are often blind to local point set subtleties. For example, the Warsaw Circle (definition below) and a point, while not of the same homotopy type, look alike in the sense that they are "weakly equivalent". Roughly, this means that the inclusion of a point into the Warsaw Circle induces an isomorphism of any of the usual algebraic invariants, homotopy, homology, etc. Recall that the Warsaw Circle consists of the subspace of the Cartesian plane consisting of the graph of $y = \sin \frac{1}{x}, 0 < x \leq 1$, the vertical interval $\{0\} \times [-1, 1]$, and an arc from $(0, 0)$ to $(1, \sin 1)$ which misses the rest of the graph and the interval. The algebraic invariants of homotopy theory tend to ignore phenomena like this where geometry meets analysis.

The geometry that homotopy theory sees well was codified in the fundamental work of J. H. C. Whitehead, in the notion of a "CW complex". Two features which make this idea successful are: (i) a weak equivalence of CW complexes is in fact a

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homotopy equivalence (an inverse up to homotopy can be constructed), and (ii) CW complexes are filtered by "skeleta" which are related by means of a construction called "attaching a cell". The first of these properties means that equivalences can be detected by homotopy groups, and the second property sets the stage for inductive arguments. Meanwhile, problems such as those presented by the Warsaw Circle have been erased from consideration. By restricting attention to the problem of classifying significantly occurring homotopy types of CW complexes, we are in a context where results can be obtained. Indeed, in 1950 at the International Congress of Mathematicians, Henry Whitehead delivered a paper entitled *Algebraic homotopy theory.* He wrote, "The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry." His indication of what had been accomplished at that time has been, until the appearance of Baues's book, a high watermark for the development of this subject in a certain direction.

The term "combinatorial homotopy" is generally understood to comprise questions of characterizing homotopy types of CW complexes in terms of algebraic invariants and having the flavor of combinatorial group theory. In fact, in low dimensions these two subjects have an extensive, overlapping history. Briefly, the content of Baues's book is a theory for 3-types and 4-dimensional CW complexes that runs parallel to the classical theory for 2-types and 3-dimensional CW complexes. Much of the material is new research, presented in this book for the first time.

To continue, we should have a definition for n-type. A space X (understood to be a CW complex herein) is an *n-type* provided that its homotopy groups are trivial in dimensions greater than n, $\pi_i X = 0,\ i > n$. Generally, n-types are infinite dimensional in the sense that the CW filtration is infinite in a nontrivial manner. Since dimension is such a natural notion, it may be worthwhile to digress to a discussion of its relation to n-type. First of all, starting with an arbitrary CW complex K and a natural number n, we can produce an n-type by attaching cells in dimensions $n+2$ and above to kill homotopy groups in dimensions $n+1$ and above. So we can introduce the notion of an associated n-type, K_n, as an n-type together with a map $K \to K_n$ inducing isomorphisms of homotopy groups π_i for $i \leq n$. This construction is known as a "homotopy section" or a "Postnikov approximation". Obstruction theory shows that K and L have equivalent n-types if and only if there is a map $f : K \to L$ inducing an isomorphism on π_i for $i \leq n$. The cellular approximation theorem then enters to show that a complex of dimension at most $n + 1$ is sufficient to capture the information contained in the associated n-type. The upshot is that while the $(n + 1)$-skeleton of K is not homotopy invariant, it does determine the homotopy invariant n-type.

One can regard the subject of combinatorial homotopy as studying the converse question of obtaining n-types from purely algebraic data. In this generality, the subject enjoys an internal tension represented on the one hand by "simplicial homotopy theory" (cf. [C]) and on the other hand by the material treated by Baues. One aspect of this tension is reflected in the size of the kinds of models in use. To gain some idea for the point of view developed in Baues's book, let us discuss some features of the theory for 1- and 2-types.

Now a 1-type is just a space X whose universal covering space is contractible (aspherical spaces in some contexts). In this case, the 1-type is completely determined by the fundamental group. Two-dimensional models for a 1-type can be obtained from a presentation of the fundamental group by generators and relations. One attaches 2-cells to a "bouquet" of circles, a circle for each generator. Attaching maps for the 2-cells are determined up to homotopy by the relator words.

Next we have the 2-types. Here the determining algebraic data consists of the fundamental group π_1, the second homotopy group π_2 regarded as a π_1-module, and the Eilenberg-Mac Lane "k-invariant" $k \in H^3(\pi_1 : \pi_2)$.

Even though it is cumbersome to work in terms of 3-dimensional models which determine the 2-type, it still may be instructive to see a construction. As before, we use generators and relations for π_1 to construct a 2-complex L. Next we join a bouquet of 2-spheres to L to form M, one sphere for each generator of π_2 (regarded as a π_1-module if we wish to have a "small" model). The k-invariant determines a surjective homomorphism $\varphi\colon \pi_2 M \to \pi_2$. Enough 3-cells are then attached to M to kill the kernel of φ. In the case where $k = 0$, we could start with L equal to the 3-skeleton of the 1-type determined by π_1 (the Eilenberg-Mac Lane complex $K(\pi_1, 1)$). Then $\pi_2 M$ is a free $Z[\pi_1]$-module and the homomorphism φ is the beginning of a presentation for π_2 as a π_1-module. Rather than models, a more tractible object for the theory of 2-types is the "crossed chain complex". In the case of a CW complex X, the crossed chain complex $\rho(X)$ consists of the following data:

$$\rho_n(X) = \pi_n(X^n, X^{n-1}), \qquad n \geq 2,$$
$$\rho_1(X) = \pi_1(X^1)$$

where X^k is the k-skeleton of the CW filtration on X. There are "boundary" maps

$$d_n : \rho_n(X) \to \rho_{n-1}(X)$$

given by composition

$$\pi_n(X^n, X^{n-1}) \to \pi_{n-1}(X^{n-1}) \to \pi_{n-1}(X^{n-1}, X^{n-1})$$

from the homotopy exact sequences. The groups ρ_n are abelian for $n \geq 3$ and are free $Z[\pi_1]$-modules on the n-cells. For $n = 2$, ρ_2 has the structure of a "free crossed-module", a notion developed by Whitehead to describe this nonabelian group with operators. The 2-type is determined by the information contained in $\rho(X)$ for $n \leq 3$.

We can now delve into the material in Baues's book. Its central novelty is the notion of a quadratic module. These are groups with operators, together with considerable extra restraints in the form of identities, satisfied by the operations and compatible with various homomorphisms. The notion of a quadratic module stands in the same relation to 3-types as the notion of crossed-module stands to 2-types. Before providing the definition, we describe its prototype in group theory. Recall that if a group G is filtered by its lower central series

$$\Gamma_1 G = G,\ \Gamma_2 G = [G, G],\ \Gamma_3 G = [G, \Gamma_2 G], \ldots$$

then there is a map

$$w\colon C \otimes C \to G/\Gamma_3 G$$

given by

$$\{x\} \otimes \{y\} \to [x, y]$$

where C is obtained by abelianizing $G/\Gamma_3 G$. In addition, there is an exact sequence

$$C \otimes C \xrightarrow{w} G/\Gamma_3 G \to C \to 0,$$

so the map w serves to measure the quadratic piece of G: If G is a free group, then the kernel of w is $\Gamma(C)$, where Γ is the quadratic functor constructed by J. H. C. Whitehead. To give a bit more detail, we turn back to crossed-modules to give some definitions. Let M and N be groups, $\partial : M \to N$ a homomorphism; and suppose N acts on M, on itself by conjugation, and ∂ is equivariant. This action defines the structure of a *crossed module* if the action satifies two other conditions, summarized in the commutative diagram:

$$\begin{array}{ccc} M \times M & = = = & M \times M \\ {\scriptstyle \partial\times 1}\downarrow & & \downarrow{\scriptstyle c} \\ N \times M & \longrightarrow & M \\ {\scriptstyle 1\times\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ N \times N & \xrightarrow{c} & N \end{array}$$

where the maps marked c are conjugation,

$$c(a, b) = a + b - a.$$

Indeed, commutativity of the lower square is equivariance of ∂. It is part of the tradition of this subject to use additive notation, since the groups are hardly ever abelian, and never mind that the term *module* is applied to something nonabelian.

If only the lower square of the diagram above commutes, the structure is called a *precrossed module*, and the difference of the two ways around the top is called the *Peiffer commutator*. This combination of identities and compatibility requirements is an example of what pervades the entire theory.

An analogue of the lower central series for groups can be defined for precrossed modules using Peiffer commutators, with a "nil (2)-module" being the analogue of a group with $\Gamma_3 = 0$. If our precrossed module is a nil (2)-module, we can obtain a map analogous to w above as follows. Let M^{cr} be M with additional relations to give a crossed module structure for the N-action. Let C be the abelianization of M^{cr}, regarded just as a group. Define

$$\omega \colon C \otimes C \to M$$

by $\{x\} \otimes \{y\} \to -x - y + x + y^{\partial x}$, where the action of N on M is denoted exponentially. The right-hand side is the Peiffer commutator, like the case for groups. We obtain the structure of a *quadratic module* when there is an additional N-group, L, and a commutative diagram of N-groups

$$\begin{array}{ccccc} C \otimes C & = = = & C \otimes C & & \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle w} & & \\ L & \xrightarrow{\delta} & M & \xrightarrow{\partial} & N \end{array}$$

with $\partial\delta = 0$ and ω lifting the map defined w above. The lift ω is required to be equivariant, where the action of N on $C \otimes C$ is given by the formula

$$(\{x\} \otimes \{y\})^\alpha = \{x^\alpha\} \otimes \{y^\alpha\}$$

where, if $x \in M$, we write $\{x\}$ for its image in C. In addition, there are two further requirements:

(i) for $a, b \in L$

$$\omega(\{\delta a\} \otimes \{\delta b\}) = [a, b],$$

the commutator in L;

(ii) the action of N on L satisfies,

$$a^{\partial x} = a + \omega(\{x\} \otimes \{\delta a\} + \{\delta a\} \otimes \{x\})$$

for $a \in L, x \in M$.

The two conditions (i), (ii) are lifts of identities satisfied by the Peiffer commutator w. In the case for spaces, one obtains the following diagram, where the quadratic complex of X is denoted $\sigma(X)$, and its structure is forced by the presence of 2-cells:

$$\begin{array}{ccccccccccc}
 & & & & C \otimes C & = & C \otimes C & & & & \\
 & & & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} & & & & \\
\longrightarrow & \sigma_4 & \longrightarrow & & \sigma_3 & \longrightarrow & \sigma_2 & \longrightarrow & \sigma_1 & \longrightarrow & \pi \\
 & \downarrow{\scriptstyle =} & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
\longrightarrow & \rho_4 & \longrightarrow & & \rho_3 & \longrightarrow & \rho_2 & \longrightarrow & \rho_1 & \longrightarrow & \pi
\end{array}$$

where the bottom row represents the crossed-chain complex of X, the non- isomorphisms are surjections, $C = \rho_2$ made abelian, the part involving w and ω is a quadratic module, and the columns are exact, among other things.

Next I will try to convey more of the content of the quadratic chain complex $\sigma(X)$ without becoming overly technical. Just as there is a free crossed module associated with a set of generators $Z \to N$, there is a free quadratic module associated with a set of generators $Z \to M$. The groups in $\sigma(X)$ (while not necessarily free) are free in the following senses. At the bottom, $\sigma_1(X) = \rho_1(X)$. The two-cells in X determine a map $Z_2 \to \rho_1(X)$ such that $\rho_2(X)$ is the free crossed module corresponding to this data. Then

$$\sigma_2(X) \to \rho_2(X)$$

is the free nil (2)-module generated by the two cells. Thus $\rho_2(X)$ is a kind of abelianization of $\sigma_2(X)$ using the Peiffer commutator. Next $\sigma_3(X)$ extends the structure so far, to a free quadratic module, and

$$\rho_3(X) \cong \sigma_3(X)/im\omega$$

under the natural map. The nontrivial way that $\sigma(X)$ extends $\rho(X)$ can be sensed from the exact sequence

$$0 \to \Delta_B \to C \otimes C \xrightarrow{\omega} \sigma_3(X) \to \rho_3(X) \to 0$$

where

$$\Delta_B = \Gamma B + [B, K] \subset \Gamma(K)$$

with $B, (K)$ representing the 2-boundaries, (2-cycles) of $\rho(X)$ and Γ is Whitehead's quadratic functor. For $n \geq 4, \sigma_n(X) = \rho_n(X)$ as free $Z[\pi]$-modules, $\pi = \pi_1(X)$.

Each of the objects $\sigma_n(X)$ is free on a set corresponding to the n-cells of X, but the generating maps for $n = 2, 3$ are not constructed directly. The difficulty is that while the cell structure prescribes generating data for $\rho(X)$, the appropriate lifts of this data in

$$\begin{array}{ccc} \sigma_2 & \longrightarrow & \rho_2 \\ & & \uparrow \\ & & Z_3 \end{array} \qquad \begin{array}{ccc} \sigma_3 & \longrightarrow & \rho_3 \\ & & \uparrow \\ & & Z_4 \end{array}$$

needed to construct directly σ_3 and the map $\sigma_4 \to \sigma_3$ are not apparent in the cell structure. Furthermore, one cannot lift these maps willy-nilly and still expect a geometrically significant result.

Instead, $\sigma(V)$ can be constructed for a simplicial set V and the existence of $\sigma(X)$ inferred from properties of $\sigma_s = \sigma(SX)$ where SX is the singular complex of X.

Three of the main theorems in the book are the following:

Theorem. *The full homotopy category of* 3*-types is equivalent to the localization of the category of quadratic modules with respect to weak equivalences.*

Theorem. *The full homotopy category of connected* 3*-dimensional CW complexes is equivalent to the homotopy category of* 3*-dimensional totally free quadratic chain complexes.*

Theorem. *The homotopy types of connected* 4*-dimensional CW complexes are in* 1–1 *correspondence with the homotopy types of* 4*-dimensional totally free quadratic chain complexes.*

Here the term "totally free" implies that each object is as free as it can be while the other usages are standard to the subject.

Computation can be made with this theory, and the text supplies many examples. The author seeks examples where the answer has interest independent of the method. To mention just one, the group of based homotopy equivalences of the 3-manifold obtained by taking the connected sum of two copies of 3-dimensional real projective space is shown to be cyclic of order two. The calculation strikes the reviewer as a good illustration of the theory in use.

The book develops its subject in six chapters. The first three review prerequisites. Chapter 3 provides a detailed and accessible account of the full theory for 2-types. Already one has probably the best introduction to this part of combinatorial homotopy theory available in textbook form. In Chapter 4, the theory of quadratic modules is developed, and most of the main theorems are proved. Chapter 5, entitled *Cohomological invariants*, generalizes work of J. H. C. Whitehead involving "Pontrjagin operations" as a system of invariants for 1-connected 4-dimensional CW complexes. Chapter 6 is entitled *The cohomology of categories and the calculus of tracks*, and I will not attempt any description of its contents.

I can assert that the book is an admirably clear presentation. There are lots of examples and plenty of illustrative calculation. The formal calculations needed to put this subject in mind are carried out in great detail, far more than would be appropriate in a research paper. The level is suitable for students with a good background in homotopy theory, as in the early chapters of [W]. Given this, the development is, for the most part, self-contained. The preparation of this review forced me to come to some kind of understanding of this strange material, so the review itself can be taken as an indication of the author's success in presenting his material. I found no typos worth passing on. The typesetting is attractive with superb layouts for diagrams.

My only cavil, and it is minor, concerns the manner in which J. H. C. Whitehead's quadratic functor Γ is introduced. There is nothing wrong, but I found the treatment too terse. I would recommend the original account [Wh] as collateral reading. For essentially the same reason, I would recommend as reading collateral to Chapter 3 the paper by Mac Lane [M]. Both papers present their topics in terms of generators and relations and (in my view) serve to put this subject more firmly in mind.

The text is preceded by a preface written by R. Brown. It does a good job of paving the way toward the new material, but I must quarrel with one statement. Brown discusses the events surrounding the introduction of the higher homotopy groups by E. Cech at the International Congress in 1932 and the subsequent withdrawal of all but a short paragraph for the Conference Proceedings. At that time several people observed that these groups were abelian. Brown writes, "On these grounds, it seems, it was felt that the groups had to be the same as the homology groups of the space." As Brown points out, the Hopf map was already known, so he finds the whole situation "curious". My understanding has been that it was the abelian nature of these groups that *alone* persuaded people at that time to think that the higher homotopy groups must miss much of the geometry, which was already understood (by J. W. Alexander among others) to require nonabelian invariants beyond the fundamental group.

In conclusion, I can warmly recommend this book both to experts and to a new generation of mathematicians who may be able to come to grips with this tantalizing but unruly subject.

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