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The geometry of four-manifolds, by S. K. Donaldson and P. B. Kronheimer. Clarendon Press, Oxford, 1990, 440 pp., $75.00. ISBN 0-19-853553-8

It is now ten years since Simon Donaldson showed how the ideas of gauge theory could be used to penetrate deeply within the differential topology of four-manifolds. During this past decade, he has not only received an array of mathematical honours, including a Fields medal for this work, but he has also advanced the subject to the extent that there are competing schools of workers currently highly active in the area in the USA, Japan, and Europe.

Now, in retrospect, it is worthwhile to recall the mathematical environment in which the subject was created in the early 80s. At that time *instantons*—self-dual solutions of the Yang-Mills equations—were familiar mathematical objects, thanks to the new-found friendship between mathematicians and physicists that grew out of this common ground over the previous five years. Instantons were indeed well known, but largely in the context in which they were introduced from physics—that of finite action solutions on $\mathbf{R}^4$ or by conformal invariance on S^4. There was some attempt to see them in a wider framework, in particular in the 1978 paper of the reviewer, Atiyah, and Singer that described the Penrose-Ward point of view and put forward the natural context of self-dual connections on self-dual four-manifolds. This could be thought of as something like a quaternionic analogue of holomorphic line bundles over Riemann surfaces, but was an idea that effectively died through lack of examples (a deficit that has recently been rectified by a theorem of Taubes). Moving away from this linkage between self-duality of the connection and manifold was itself a bold step.

There were nevertheless results that provided a justification for doing so. One of these was the viewpoint of symplectic geometry. Atiyah and Bott were using this aspect to explain, in terms of Yang-Mills theory on two-manifolds, the old results of Narasimhan and Seshadri on stable holomorphic bundles over Riemann surfaces. An analogous formalism, as Donaldson himself noticed early on, led to gauge-theoretic equations (now called the Hermitian-Einstein equations) whose solutions might be expected to exist on stable bundles over higher-dimensional Kähler manifolds. The instanton connections lifted to the Penrose twistor space $\mathbf{CP}^3$ furnished explicit examples and the same equations on a Kähler manifold of two complex dimensions were precisely the same as the anti-self-dual Yang-Mills equations. The self-duality of the conformal structure was irrelevant here, but the geometrical context was equally natural. Another result, perhaps more important for Donaldson's work, was the 1982 paper of Taubes which showed how one could produce self-dual connections on a non-self-dual four-manifold by the technique now known as "grafting". Adding in the analytical results of Uhlenbeck, the right atmosphere existed for a breakout from the confines of the original instanton problem into a more general geometrical theory, specifically tailored to four dimensions, and supported by enough examples to be able to gain an intuitive feel for the possibilities. The fact that the instantons not only *existed* on rather general four-manifolds but could also be used to analyse their differential topology was something that few could have predicted, and it is Donaldson's achievement that so many interesting results have come out of the subject.

In some sense, gauge theory applied to differential topology is a nonlinear, nonabelian version of Hodge theory. Ordinary Hodge theory describes the topologically invariant cohomology groups on a compact Riemannian manifold by harmonic forms—solutions to a linear Laplace equation. It is particularly effective, as Hodge himself showed, when the underlying manifold is a Kähler manifold, in particular a projective variety. Gauge theory takes another metric-dependent equation, the self-dual Yang-Mills equation, and studies its space of solutions modulo equivalence, the *moduli space*. Through this analogy we are encouraged to think of this space as comparable to the vector space which is the cohomology group. That may seem natural enough, except that the moduli space is a much more complicated object. When it exists, it truly depends on the metric; it is not in general even a manifold but is a singular noncompact space. One needs a lot of convincing to believe that it contains any differential topological information at all, and it is perhaps fortunate for us that Donaldson's first theorem in the subject proved this fact in such a spectacular manner. Its importance was amplified by the parallel (but entirely different) results of Freedman, which, when combined with those of Donaldson, gave such remarkable results as the existence of exotic differentiable structures on $\mathbf{R}^4$.

This one result may at the same time have revealed the topological significance of gauge theory and also may have blinded some mathematicians to its other possibilities. Looked at from one point of view, Donaldson's first theorem only just works. In fact, with less blinkered eyes, the methods just in their infancy then were the first points in an agenda that is still rapidly developing. Among the most concrete products of the theory are the *Donaldson polynomials.* These are tangible differential invariants that can often be computed in some form and are now the everyday language of gauge theorists in topology. And these, essentially, are what the book under review is about.

The book is based on Oxford graduate lectures from 1985–1986, and the authors' aim is to construct and describe moduli spaces of instantons and to define and calculate examples of Donaldson polynomials. In so doing, they provide many of the basic results and tools for working in this area. Some of these are a distillation of published results, but others are new approaches that can only be found in this book. As an example, a particularly interesting sidelight is provided by a chapter on Fourier transforms and the place of the ADHM construction of instantons and the Mukai correspondence of sheaves on abelian varieties within that setting.

There is much more in this book than a collection of results and theorems. The reader will learn to appreciate the flexible approach as well as the methods. The mathematics is not didactic—one is not being taught a single correct way of doing things, only one of a number of possibilities. Often the authors will finish a proof with a reason why they chose one line of attack, purely in the context of what they wish to achieve in the book. One has the feeling that one is invited to try other methods too. Perhaps there is something quintessentially English in this pragmatic approach, which seems to permeate the subject, at least as presented here. Too many branches of mathematics are involved for the practitioner to adopt a single, fixed perspective.

On the other hand, there is no disguising the complexity of the subject. It is not an easy read and the mathematics is very dense. There is, however, plenty of motivation. There are many levels at which one may view the Donaldson invariants. All are presented, but the reader is always warned when he or she is being presented with an artificially simplified version. At the most naive level, the polynomials are obtained by taking certain universal cohomology classes in the space of *all* connections modulo gauge equivalence, restricted to the moduli space of self-dual (or here anti-self-dual) connections and evaluated on a fundamental class. Defining that class presents problems. In the first place, the moduli space may not be smooth. There are two causes for this and one may be removed by choosing a generic metric, with the remaining singularities being well understood in terms of reducible connections. The space is still not compact, though, and must be compactified by adding on the ideal connections with "delta-function" Yang-Mills densities which sequences of connections degenerate according to Uhlenbeck's theorem. This compactified object is still not a manifold but for a generic metric does possess a fundamental homology class.

An invariant can actually be calculated by using the theorem of Donaldson (and also Uhlenbeck and Yau) that proves the result conjectured in 1980: Stable holomorphic bundles on an algebraic surface are in one-to-one correspondence with anti-self-dual connections modulo gauge equivalence. This result yields the moduli space as a function of the complex structure and not of the metric and can in principle be found in one form or another. But problems abound, not the least of which are that Kähler metrics are not generic and that the moduli space thus produced may be a singular or nonreduced algebraic variety.

None of these problems is skirted in the book, and the reader is always offered a choice—a full proof of a special situation or an outline and reference to the literature for something more general. He or she is getting a full exposure to the subject and one that will undoubtedly provoke much contemplation (especially when the text reads "A moment's thought will show..."!). At the end of the day, however, reasons are usually given for methods or estimates. Geometry lies behind a particular choice of Sobolev norms, and guided experience dictates the appropriate size of error terms introduced by cutoffs.

Perhaps the most remarkable feature of the whole subject is the malleability that a mind with the right intuition and available techniques can impose on objects that one feels are much more rigid. In the linear Hodge theory, we are aware of alternative descriptions of cohomology groups by de Rham, singular or Čech cocycles, and can see how these adapt to deformation. For the Yang-Mills moduli space there is no such alternative. Nowhere is this more apparent than in the chapter on excision and glueing, an important set of ideas for applications.

The authors are both workers at the forefront of research in this field, and they have done the mathematical world a great service by devoting time to produce the book. It is authoritative, comprehensive, and yet written with the reader in mind. It must be regarded as compulsory reading for any young researcher who is approaching this difficult but fascinating area. Of course, new results and points of view have descended on the subject since the book was written, and the authors make no claim to be exhaustive. Some results have changed our intuition about four-manifolds. At one stage, the underlying differentiable structures of algebraic surfaces were thought to be more fundamental than appears to be the case now. Others have come again from mathematical physics, in particular, as the ideas of Witten on topological quantum field theory have proven their worth in three dimensions. A four-dimensional analogue, involving Floer groups, is a topic of much current research. Work is in progress on the application of gauge theory to the geometry of a four-manifold with a fixed embedded surface in it, thus answering some old questions of Thom. These are all topics in which the authors have a great deal of experience and expertise. One can only hope that the lectures that current Oxford graduate students are experiencing will eventually surface in a form similar to *The geometry of four-manifolds.*

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Perturbation methods, by E. J. Hinch. Cambridge University Press, Cambridge and New York, 1991, xi + 160 pp., $59.50. ISBN 0-521-37310-7

Differential equations can be divided into those that can be solved and those that cannot. The first class is nearly exhausted by a sophomore "cookbook" differential equations course. The study of the second class is roughly divided into three approaches: qualitative, numerical, and asymptotic. The qualitative approach (which, at least for initial value problems, more or less coincides with "dynamical systems theory") gives up the attempt to find solutions and instead seeks to describe the behavior of the solutions. (In many cases this is what one wants the solutions for anyway.) The numerical approach obtains approximate solutions in the form of tables or graphs. The asymptotic approach, otherwise known as perturbation theory, also looks for approximate solutions but obtains them as formulas. When these formulas are simple enough to comprehend, they can reveal a great deal about the solution. At the simplest level, for instance, an approximate formula for a periodic solution can immediately show the influence of each variable upon the period and amplitude. At a deeper level, it is in the very nature of an asymptotic solution that its terms are sorted into orders of importance. This forces the mathematician into a style of thinking that is reminiscent of pragmatic common sense: when faced with a complicated problem, one asks which features of the problem are most important and attempts to incorporate them into the solution first. In this way, guesswork (more politely known as heuristics) comes to play an important role in the construction of solutions, and proofs of validity (that is, proof of error bounds) get pushed to the end. Often, because the mind-set needed for heuristics differs from that needed for proofs, the proofs get ignored altogether. For many authors, if a solution "looks" asymptotic (that is, if it is "uniformly ordered", as defined below) and agrees well enough with numerical solutions, then it is good enough.

Is it good enough? Let us briefly examine what may be the paradigm case, the case that gave rise to the name "perturbation theory". Everyone knows