logarithmic gauges and the superiority of the alternate (but equally heuristic) method called "matching in an overlap domain" in this situation. There is a short derivation of the asymptotic approximation to the limit cycle of a Van der Pol oscillator in the relaxation case, a complicated problem involving several different matchings between different domains. There is an example of the use of the WKB method to study an exponentially small term in a boundary layer problem, which leads to paradoxical results when it is treated by matching alone.

As these examples indicate, the strengths of the book lie in the area of boundary layers and matching. There is a short chapter on asymptotic evaluation of integrals. There is some, but very little, attention to the third major area within perturbation theory, namely, nonlinear oscillations. In contrast to the boundary layer problems, all of the oscillatory problems treated in this book are extremely simple: Duffing's equation is solved by Lindstedt's method, the Van der Pol equation (in the nearly linear case, the opposite of the relaxation case mentioned above) is solved by multiple scales, and a nonlinear wave equation is solved by an averaged Lagrangian. Aside from the last problem there is no mention of averaging, which is probably the most useful method in nonlinear oscillations. (Nonlinear oscillations include celestial mechanics, from which, as mentioned above, perturbation theory took its name; there are no celestial mechanics included in this book.)

It is not entirely clear for which audience the book is intended; it would seem to be difficult to find a reader who knows most of the things the author assumes but not most of the things that he says. The book is much too short and sketchy and hurries too rapidly into difficult examples to serve as an introduction. Yet it does treat elementary topics, which would not be required by an advanced reader. Most of the difficult examples are treated so briefly that they are best regarded as exercises with ample hints. For the Van der Pol example, the reader is expected to know immediately that $K_{1/3}$ and $I_{1/3}$ are Bessel functions (they are never identified), to know their asymptotic properties, and to know how they are used to solve Airy equations; all of this is passed over in one line. An acquaintance with fluid mechanics is also assumed. The book will probably find its greatest usefulness as a reference book for those with considerable background, but, for this purpose, one would wish for a better bibliography.

JAMES MURDOCK
IOWA STATE UNIVERSITY
E-mail address: murdock@pollux.math.iastate.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, April 1993

0273-0979/93 $1.00 + $.25 per page

The Stefan problem, by Anvarbek Meirmanov (translated from the Russian by M. Neizgodka and Anna Crowley). Walter de Gruyter, Berlin, 1992; ix+244 pp., $89.00. ISBN 3-11-011479-8

In crude terms, the Stefan problem is that of solving $\partial/\partial t[u + H(u)] = \Delta u$ where u is the temperature of some material and H is the Heaviside function.

It has a long history of mathematical activity and quiescence. Put very briefly, the progenital problem was posed in 1831 by Lamé and Clapeyron and restated by Stefan in 1889; the weak formulation had its origins in the 1940s and was solved by Oleinik and Kamenomostkaja in 1960; and the one-dimensional classical formulation was solved and narrated by Rubinstein in 1971. This left the major objective of proving the local existence of the multidimensional classical problem to be attained by Professor Meirmanov in 1979, although a more recondite proof was given by Hanzawa in 1981. Even Professor Meirmanov's proof was sufficiently intricate for many Western observers to remain skeptical until an inquisition, with a happy outcome, was held in Maubisson, France, in 1984! The breakthrough came with the choice of a regularising mechanism that was both tractable and could be made arbitrarily small. Professor Meirmanov's choice was to introduce the surface Laplacian of the coordinate normal to the interface $u = 0$ into the Stefan condition of energy conservation at this interface, this choice being greatly facilitated by the use of what he terms Von Mises variables which conform to the interface. This regularisation has no physical basis, but it is tantalisingly close, although technically very different, from the mechanism that is often assumed to be the most prevalent in the real world, namely, that of introducing surface energy. This is done by inserting the Gibbs-Thomson term, which is proportional to the mean curvature, into the *equilibrium* boundary condition rather than the energy conservation condition. Another candidate would be the insertion of the normal velocity into the equilibrium condition. Indeed, the mathematical treatment of these latter problems is still incomplete even though major strides have been made within the past two years by Luckhaus and Almgren.

The reason why the Gibbs-Thomson condition is of such abiding interest outside the mathematics community is that its effect on Stefan problems of most practical interest is to engender all the fascinating mechanical and geometric complexity associated with dendritic growth cases where the Stefan problem is *supercooled.* (Put crudely, in the case where the liquid is supercooled and the solid is superheated, this means we are trying to solve the problem $\partial/\partial t[u - H(u)] = \Delta u)$. This highlights my only serious criticism of Professor Meirmanov's book: there is almost no mention of the mathematical difficulties encountered in modelling supercooling. It would have been most helpful to see the author's views on the comparative applicability of various candidate models and regularisation methods for this very difficult problem.

Given that the author was going to restrict himself to unsupercooled problems, the layout of the book is admirably clear. It begins with the relevant background analysis and then describes the principal local results in some thirty pages. There then follows the extension to longer time intervals for a restricted class of solutions using the so-called Duvaut transformation, whereby the temperature is integrated with respect to time. The one-dimensional problem falls into this restricted category, and a special chapter is devoted to it and related porous medium flows.

One of the most interesting aspects of the Stefan problem is the interplay between the weak and classical solutions. As always, classical solutions are weak solutions, but, as distinct from elliptic or hyperbolic problems, here there are situations where both classical and weak solutions are possible but differ from each other. This is most strikingly the case when so-called mushy regions exist, and these are described in detail in a later chapter, although no real mention is made of their physical interpretation. However, the book ends with a change of emphasis, with descriptions of time-periodic solutions (as might occur, say, with thermostats), approximate solutions to some ingot solidification problems, and some joint work on the wide-open question of alloy solidification, which leads inevitably to the study of vector Stefan problems.[1] The final pages contain some enigmatic statements about the thermodynamic basis for theories of alloy solidification.

I hope it is clear from the above that this book will be extremely valuable to all mathematicians working in free-boundary problems because it collects all the seminal work in the area carried out by the author over the past fifteen years. However, the book does not purport to provide an overview of the subject, for which many conference proceedings or the book of Crank (not even mentioned here) should be consulted. The first-ever English text on the Stefan problem by Rubinstein does receive brief mention, but I was saddened to see a one-dimensional theory described without references to that pioneering work. Indeed, it would have been helpful if the subjectiveness of the book had been stressed in the introduction.

Notwithstanding these criticisms of style, Professor Meirmanov and his translators, Marek Niezgodka and Anna Crowley, are to be congratulated on having produced this admirable record of the mathematical heart of the Stefan problem.

JOHN OCKENDON
MATHEMATICAL INSTITUTE, OXFORD UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, April 1993

0273-0979/93 $1.00 + $.25 per page

Hardy-type inequalities, by B. Opic and A. Kufner. Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Longman Group UK Ltd., Harlow, 1990, vii + 333 pp., $44.00. ISBN 0-582-05198-3

In 1920 when G. H. Hardy discovered the inequality

$$\text{(1)} \qquad \left\{\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\,dt\right|^p dx\right\}^{1/p} \le \frac{p}{p-1}\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p},$$

$1 < p < \infty$, in an attempt to simplify the proofs of Hilbert's double series theorem, he could hardly have foreseen the profound influence this inequality and its variants and generalizations would have on the development of many areas in analysis. In Fourier analysis, for example, it is the key factor in the proof of the Hardy-Littlewood maximal theorem; and the proof of the Marcinkiewicz theorem on the interpolation of operators requires, in a significant way, only a

[1]A new kind of vector Stefan problem has recently gained prominence in the macroscopic theory of superconductivity.