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BOOK REVIEW

Interpolation of rational matrix functions, by J. A. Ball, I. Gohberg, and L. Rodman. Operator Theory: Advances and Applications, vol. 45, Birkhäuser Verlag, Basel, 1990, 605 pp., $129.00. ISBN 3-7643-2476-7

The area

Interpolation theory of functions has a long and interesting history, which, roughly speaking, may be divided into three periods. The classical period, which has its roots deep in the two previous centuries, mainly concerns scalar functions and involves contributions of mathematicians such as Lagrange, Sylvester, Hermite, Schur, Carathéodory, Toeplitz, Nevanlinna, Pick, Takagi, Loewner, and Nehari. During this first period most of the work took place in a complex function theory setting; methods of functional analysis and operator theory were introduced by N. I. Akhiezer and M. G. Kreĭn later in the thirties. The second period begins after World War II and concerns matrix- (and operator-) valued functions. Papers of Sz.-Nagy and Koranyi [32, 33] were among the first major contributions. Many different approaches were invented with methods of operator theory in a prominent position, and a rich theory was developed involving the work of many outstanding mathematicians. First rate examples are the commutant lifting approach by Sarason [30] and Sz.-Nagy and Foias [31] in the sixties; the publications of the Potapov school (see, e.g., [27]); and the series of papers by Adamjan, Arov, and Kreĭn [1–5] in the seventies (see [29, p. 45] for a more detailed survey).

In the early eighties the third period starts with the discovery that engineering problems of the type appearing in mathematical system theory, in particular, in control theory with an H-infinity optimality criterion, can be reduced to interpolation problems for matrix-valued functions of which the entries are rational (i.e., quotients of polynomials). Glover's 1984 paper [16], which solved the H-infinity model reduction problem in the time domain by using the Adamjan-Arov-Kreĭn solution of a vector-valued Nehari-type extension problem, was one of the first results in this direction. This development led to a new field of research in which the emphasis is on rational matrix functions and on the search for explicit formulas of interpolants in a form which is suitable for control applications. Both the special interest in rational functions and the demand for explicit formulas came from the engineering side. At the same time, system and control theory provided a base for developing new tools that allowed one to solve rational matrix function problems by linear algebra methods involving finite matrices only.

The book

The present book belongs to the third period; it develops an interpolation theory for rational matrix functions entirely in a system theory state space framework. The precise meaning of the latter phrase will be explained later in this review. First a few words about the type of problems considered. The book treats homogeneous problems (when any nonzero scalar multiple of one solution is again a solution) and nonhomogeneous problems, both with and without additional metric or norm constraints. Included are the rational matrix analogues of the classical scalar interpolation problems of Lagrange, Lagrange-Sylvester, Nevanlinna-Pick, Carathéodory-Toeplitz, Schur-Takagi, and Nehari, which are nonhomogeneous in general, and the matrix analogue of the problem of constructing a rational function with prescribed zeros and poles, which is a homogeneous problem and has many different faces in the matrix case. Interpolation is understood in a rather broad sense. For instance, the problem of finding a matrix polynomial with given remainders after division by various other given polynomials is treated here as a nonhomogeneous interpolation problem. A representative sample of some of the H-infinity engineering problems is also presented.

An example of the type of interpolation questions treated in the book is the following problem, which appeared in the literature in a somewhat more general form in the beginning of the seventies (see [15]). Find (if possible) a rational $m \times r$ matrix function F that is analytic at infinity and satisfies the following conditions:
(TNP1) F has all its poles in $\operatorname{Re}\lambda < 0$;
(TNP2) $\|F(\lambda)\| < 1$ for $\operatorname{Re}\lambda \geq 0$;
(TNP3) $x_i F(z_i) = y_i$, $i = 1, \dots, N$;
(TNP4) $F(w_j) u_j = v_j$, $j = 1, \dots, M$;
(TNP5) if $z_i = w_j$, then $x_i F'(z_i) u_j = \rho_{ij}$.
Here $z_1, \dots, z_N$ and $w_1, \dots, w_M$ are given points in $\operatorname{Re}\lambda > 0$ such that $z_i \neq z_j$ and $w_i \neq w_j$ whenever $i \neq j$. Furthermore, $x_1, \dots, x_N$ and $u_1, \dots, u_M$ are given nonzero elements of $\mathbb{C}^{1\times m}$ and $\mathbb{C}^{r\times 1}$, respectively; $y_1, \dots, y_N$ and $v_1, \dots, v_M$ are given vectors in $\mathbb{C}^{1\times r}$ and $\mathbb{C}^{m\times 1}$, respectively; and ρ_{ij} are given complex numbers. The norm constraint in (TNP2) is of the type appearing in the classical Nevanlinna-Pick problem. Conditions (TNP3) and (TNP4) specify values of $F(\cdot)$ along directions on the left-hand and right-hand sides, respectively; and therefore, one refers to the problem (TNP1)–(TNP5) as a *bitangential Nevanlinna-Pick interpolation problem*.

Entirely in a system theory state space framework

The method employed by the authors is called the *state space method* (cf., [11]). It has its roots in the mathematical system theory of the sixties (see [26] or [24]) and is based on the idea of state space realization. Recall that in the Kalman approach an input-output system is a set of differential equations of the form

$$
\begin{aligned}
x'(t) &= Ax(t) + Bu(t), \qquad x(0) = 0, \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\tag{1}
$$

where A is a square (complex) matrix of order n, say, which acts on the *state space* $\mathbb{C}^n$, and B, C, and D are matrices of sizes $n \times r$, $m \times n$, and $m \times r$, respectively. The system is assumed to be at rest at time $t = 0$; and hence, by applying the

Laplace transform, one sees that in the so-called frequency domain the connection between input $u(\cdot)$ and output $y(\cdot)$ is given by $\hat{y}(\lambda) = F(\lambda)\hat{u}(\lambda)$, where

$$F(\lambda) = D + C(\lambda I_n - A)^{-1}B\,. \tag{2}$$

Here, I_n is the $n \times n$ identity matrix. The matrix function $F(\cdot)$ is called the *transfer function* of the system (1), and one calls (2) a (*state space*) *realization* of $F(\cdot)$. From (2) it is clear that $F(\cdot)$ is an $m \times r$ rational matrix function which is analytic at infinity. A first basic fact from system theory, which is known as the *realization theorem* (see [26]), states that conversely any rational matrix function which is analytic at infinity may be realized as the transfer function of a system of the type (1); in other words, any such function $F(\cdot)$ can be written in the form (2). The realization (2) is said to be *minimal* if the order n of the state matrix A in (2) is as small as possible. Minimal realizations are unique up to a *state space similarity*; that is, if (2) is a minimal realization of $F(\cdot)$, then any other minimal realization of $F(\cdot)$ is obtained by replacing A, B, and C by $S^{-1}AS$, $S^{-1}B$, and CS, respectively, where S is an arbitrary nonsingular matrix. These facts allow one to deal with rational matrix functions in terms of matrices, and in the book they are used to treat interpolation problems in a linear algebra context.

Null-pole-triples

As a first step in the direction of interpolation, let us see in the state space setting what is meant by the null-pole structure of a regular $p \times p$ rational matrix function Θ. Here, regular means that $\det\Theta$ does not vanish identically. We begin with the poles, i.e., with points λ_0 in $\mathbb{C}$ where

$$\Theta(\lambda) = \sum_{j=-q}^{\infty} (\lambda - \lambda_0)^j \Theta_j$$

has a nontrivial singular part $\sum_{j=-q}^{-1} (\lambda - \lambda_0)^j \Theta_j$. Fix a subset σ of $\mathbb{C}$. The sum of the singular parts of the poles of Θ in the set σ is a rational matrix function which is analytic at infinity and has the value 0 at infinity, and hence it admits a minimal realization of the form $C(\lambda - A)^{-1}\widetilde{B}$. Any pair (C, A) arising in this way is called a *right pole pair* of Θ relative to σ. By the state space similarity theorem for minimal systems, pole pairs are unique up to similarity, i.e., any right pole pair of Θ relative to σ is of the form $(CS, S^{-1}AS)$, where S is an arbitrary nonsingular matrix and (C, A) is as above. For the null structure, $\Theta(\cdot)^{-1}$ is employed. A pair of matrices (A, B), where A is $n \times n$ and B is $n \times p$, is called a *left null pair* of Θ relative to σ if (A, B) is a left pole pair of Θ^{-1} relative to σ; that is, there exists a $p \times n$ matrix $\widetilde{C}$ such that $\widetilde{C}(\lambda - A)^{-1}B$ is a minimal realization of the sum of the singular parts of Θ^{-1} in σ.

In contrast with the scalar case, a rational matrix-valued function may have a pole and zero at the same point; for example,

$$\Theta(\lambda) = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}, \qquad \Theta(\lambda)^{-1} = \begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & 1 \end{pmatrix} \tag{3}$$

both have an entry with a pole at zero. It follows that, in general, the null structure and pole structure of Θ on σ are coupled. This additional information may be encoded in the *null-pole subspace* which is defined as

$$\Theta\mathcal{R}_{p\times 1}(\sigma) = \{\Theta h | h \in \mathcal{R}_{p\times 1}(\sigma)\},$$

where $\mathcal{R}_{p\times 1}(\sigma)$ denotes the set of all rational $\mathbb{C}^{p\times 1}$-valued vector functions having no pole in σ. It turns out that given a left null pair (A_ζ, B_ζ) and a right pole pair (C_π, A_π) of Θ, both relative to σ, there exists a unique $n_\zeta \times n_\pi$ matrix Γ (where n_ζ is the order of A_ζ and n_π is the order of A_π) such that

$$\Theta\mathcal{R}_{p\times 1}(\sigma) = \{C_\pi(\lambda - A_\pi)^{-1}x + h(\lambda) | x \in \mathbb{C}^{n_\pi},\ h \in \mathcal{R}_{p\times 1}(\sigma) \text{ is such that } \Gamma x \text{ is the sum of all the residues of } (\lambda - A_\zeta)^{-1}B_\zeta h(\lambda) \text{ in } \sigma\}\,.$$

The quintet $\{C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma\}$ is called a (*left*) *σ-null-pole-triple* for Θ. If Θ is given by a minimal realization, $\Theta(\lambda) = D + C(\lambda - A)^{-1}B$, with D invertible, then

$$\Theta(\lambda)^{-1} = D^{-1} - D^{-1}C(\lambda - A^\times)^{-1}BD^{-1}, \qquad A^\times := A - BD^{-1}C,$$

and a σ-null-pole triple for Θ may be constructed directly in terms of the matrices A, B, C, D, and $A^\times$ by using elementary spectral theory. Indeed, take P to be the Riesz projection of A corresponding to the eigenvalues of A in σ, let $P^\times$ be the corresponding projection for $A^\times$ in place of A, and put

$$\tau = \{C|_{\operatorname{Im} P}, A|_{\operatorname{Im} P}; A^\times|_{\operatorname{Im} P^\times}, P^\times BD^{-1}; P^\times|_{\operatorname{Im} P}\}\,.$$

Then τ is a σ-null-pole triple of Θ, and up to similarity any σ-null-pole triple of Θ is of this form.

Null-pole triples $\{C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma\}$ are a recent invention. Null pairs appear for the first time (under the name of Jordan pairs) in the Gohberg-Lancaster-Rodman analysis of matrix polynomials (see [18–20], and [21]). For interpolation problems, it is important to know that such pairs may also be constructed in terms of null chains and null functions, which have a longer history and originated in the work of Keldysh [25] on completeness of eigenvectors and generalized eigenvectors for analytic functions of which the values are Fredholm operators acting on an infinite-dimensional Hilbert space. Pole vectors and generalized pole vectors have their roots in the Gohberg-Sigal paper [22]. The coupling matrix Γ, which satisfies the Sylvester equation

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi,$$

appears for the first time in 1987 in the Ball-Ran papers [9, 10]; a predecessor may be found in [17]. The importance of null-pole triples follows among others from the following result. Two regular rational matrix functions Θ_1 and Θ_2 have the same σ-null-pole triple if and only if $\Theta_1(\lambda) = \Theta_2(\lambda)E(\lambda)$ for a rational matrix function $E(\cdot)$ such that both $E(\cdot)$ and $E(\cdot)^{-1}$ have no pole in σ.

An illustrative example

How does the book use the above machinery to deal with interpolation problems? Let me illustrate this on the bitangential Nevanlinna-Pick interpolation problem (TNP1)–(TNP5) mentioned above. The analysis starts with some heuristic arguments. From earlier experience—for instance, from the classical Nevanlinna paper [28]—one expects that if a solution of the problem (TNP1)–(TNP5) exists, the set of all solutions F will be parametrized by a linear fractional map,

$$F(\lambda) = (\Theta_{11}(\lambda)G(\lambda) + \Theta_{12}(\lambda))(\Theta_{21}(\lambda)G(\lambda) + \Theta_{22}(\lambda))^{-1}, \tag{4}$$

where the free parameter G is an arbitrary $m \times r$ rational matrix function which satisfies the conditions (TNP1), (TNP2). Since we want $\|G(\lambda)\| < 1$ for $\operatorname{Re}\lambda \geq 0$ to imply $\|F(\lambda)\| < 1$ for $\operatorname{Re}\lambda \geq 0$, it is natural to assume that the coefficient matrix $\Theta(\cdot) = (\Theta_{ij}(\cdot))_{i,j=1}^2$ is a rational matrix function which is J-unitary with respect to the imaginary axis, where J is the $(m+r)\times(m+r)$ signature matrix $I_m \oplus (-I_r)$, i.e., one expects Θ to satisfy

$$\Theta(-\overline{\lambda})^* \begin{pmatrix} I_m & 0 \\ 0 & -I_r \end{pmatrix} \Theta(\lambda) = \begin{pmatrix} I_m & 0 \\ 0 & -I_r \end{pmatrix}. \tag{5}$$

By taking $G(\lambda) \equiv 0$ in (4), one sees that $\Theta_{12}(\cdot)\Theta_{22}(\cdot)^{-1}$ has to be a solution of (TNP1)–(TNP5), and hence $\Theta_{12}(\cdot)\Theta_{22}(\cdot)^{-1}$ must have all its poles in $\operatorname{Re}\lambda < 0$. The latter is achieved by requiring that Θ is J-inner, i.e., Θ is J-unitary and

$$\Theta(\lambda)^* \begin{pmatrix} I_m & 0 \\ 0 & -I_r \end{pmatrix} \Theta(\lambda) \leq \begin{pmatrix} I_m & 0 \\ 0 & -I_r \end{pmatrix}, \qquad \operatorname{Re}\lambda > 0 \text{ and } \lambda \text{ not a pole}. \tag{6}$$

Note that (4) implies

$$\begin{pmatrix} F(\lambda) \\ I \end{pmatrix} = \Theta(\lambda) \begin{pmatrix} G(\lambda) \\ I \end{pmatrix} (\Theta_{21}(\lambda)G(\lambda) + \Theta_{22}(\lambda))^{-1},$$

and hence (TNP3) will be fulfilled if Θ has a zero at s_i with left tangential zero row vector $(x_i \quad -y_i)$. For (TNP4) an analogous result holds. To make a precise statement also involving the condition (TNP5), use the interpolation data in (TNP1)–(TNP5) to introduce the following matrices:

$$A_\zeta = \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_N \end{pmatrix}, \qquad B_+ = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \qquad B_- = -\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix},$$

$$A_\pi = \begin{pmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_M \end{pmatrix}, \quad C_- = (u_1 \quad u_2 \cdots u_M), \quad C_+ = -(v_1 \quad v_2 \cdots v_M),$$

$$S = (s_{ij})_{i=1,\,j=1}^{N,\ M}, \qquad s_{ij} = \frac{-x_i v_j + y_i u_j}{z_i - w_j} \quad (z_i \neq w_j), \qquad s_{ij} = \rho_{ij} \quad (z_i = w_j),$$

and put

$$\tau := \left\{ \begin{pmatrix} C_+ \\ C_- \end{pmatrix}, A_\pi; A_\zeta, (B_+ \quad B_-); S \right\}. \tag{7}$$

It turns out that the problem (TNP1)–(TNP5) is solvable if and only if one can find an $(m+r)\times(m+r)$ rational matrix function Θ, which is nonsingular on the imaginary axis, is J-inner (i.e., formulas (5) and (6) hold), and has the quintet τ in (7) as its right half plane null-pole triple. Furthermore, in that case, all solutions F of (TNP1)–(TNP5) are of the form (4), where $(\Theta_{ij}(\lambda))_{i,j=1}^2$ is the 2×2 block matrix partitioning of $\Theta(\lambda)$ according to the decomposition $\mathbb{C}^{m+r} = \mathbb{C}^m \oplus \mathbb{C}^r$.

Noticeable features

The previous result is typical for the approach used in the present book. It holds in a much wider context for various other interpolation problems, and it reduces these interpolation problems to a problem of finding a coefficient matrix of a linear fractional map which, in general, is a rational matrix function satisfying certain symmetry conditions and having a certain prescribed null-pole triple. The book develops all the machinery to deal with problems of the latter type. Typically, the solutions are sought as transfer functions in realized form. The uniqueness of minimal realizations up to similarity often provides a hint for the equations which have to be solved. For example, to find an $(m+r)\times(m+r)$ rational matrix function Θ, which is nonsingular on the imaginary axis, satisfies (5) and (6), and has the quintet τ in (7) as its right half plane null-pole triple, one has to solve the following Lyapunov equations:

$$\begin{aligned} S_1 A_\pi + A_\pi^* S_1 &= C_-^* C_- - C_+^* C_+, \\ S_2 A_\zeta^* + A_\zeta S_2 &= B_+ B_+^* - B_- B_-^*, \end{aligned} \tag{8}$$

which one can do explicitly because the matrices A_π and A_ζ have their eigenvalues in the open right half plane. Then a solution Θ exists if and only if the matrix

$$\Lambda = \begin{pmatrix} S_1 & S^* \\ S & S_2 \end{pmatrix}$$

is positive definite. Furthermore, in that case, a Θ with the desired properties is explicitly given by

$$\Theta(\lambda) = \begin{pmatrix} I_m & 0 \\ 0 & I_r \end{pmatrix} + \begin{pmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{pmatrix} \begin{pmatrix} (\lambda - A_\pi)^{-1} & 0 \\ 0 & (\lambda + A_\zeta^*)^{-1} \end{pmatrix} \Lambda^{-1} \begin{pmatrix} -C_+^* & C_-^* \\ B_+ & B_- \end{pmatrix}.$$

For the case considered here, it is straightforward to solve the two equations in (8). In fact, both S_1 and S_2 are Pick matrices, namely,

$$S_1 = \left(\frac{u_i^* u_j - v_i^* v_j}{w_i + \overline{w}_j} \right)_{i,j=1}^{M}, \qquad S_2 = \left(\frac{x_i x_j^* - y_i^* y_j}{z_i + \overline{z}_j} \right)_{i,j=1}^{N}.$$

Hence, the $(N+M)\times(N+M)$ matrix Λ is explicitly given; and to see whether or not the interpolation problem (TNP1)–(TNP5) is solvable, one has to check only that the eigenvalues of Λ are positive.

Recent developments [6–8] suggest that the approach sketched above has a much wider range. Indeed, if one replaces the input-output system in (1) by a time-varying system,

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t), \qquad x(0) = 0, \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{9}$$

and rational matrix functions by input-output operators of time-varying systems of the type (9), then many of the interpolation problems discussed in the present book have a natural time-varying nonstationary analogue (see also [12]). The interesting fact is that an appropriate time-variant version of the state space method can be used to produce solutions in much the same way as they are constructed in the present book.

More about the book

The book has six parts and an appendix about Sylvester, Lyapunov, and Stein equations. The first part (which consists of about 200 pages, a little less than one third of the book) introduces zeros and poles, null vectors and pole vectors, generalized null vectors and generalized pole vectors, chains of generalized null vectors and chains of generalized pole vectors, and builds in this way step by step the null structure and the pole structure and the null-pole triples referred to above. Here also appear the solutions of homogeneous interpolation problems which have a certain (local or global) null-pole triple as the given data. Homogeneous interpolation problems with other forms of local data (for example, problems based on divisibility) are treated in Part 2 (33 pages). The description of null-pole triples in terms of the null-pole subspace (which we used in this review) appears in Part 3 (78 pages), which treats various subspace interpolation problems. Here one also finds a rational matrix-valued version of the Beurling-Lax invariant subspace representation theorem and its recent Ball-Helton version in spaces with an indefinite metric. Part 4 (61 pages) deals with nonhomogeneous interpolation problems for rational matrix functions without metric constraints like those of Lagrange and Lagrange-Sylvester. Also, the partial realization problem from mathematical systems theory is treated here as a nonhomogeneous unconstraint interpolation problem. The rational matrix analogues of the classical scalar nonhomogeneous interpolation problems with metric constraints of Nevanlinna-Pick, Carathéodory-Toeplitz, and Nehari (including the problem (TNP1)–(TNP5) discussed above and its more general version, which is called the Takagi-Nudelman problem) appear in Part 5 (124 pages). In general, the interpolation data are of rational type. Singular or degenerate cases are not treated. The final part (55 pages) is entirely devoted to problems of H-infinity control. Included are (the so-called one-block versions of) the problems of sensitivity minimization, model reduction, and robust stabilization. For these problems the engineering motivation is described, and the reduction to interpolation problems of the type appearing in Part 5 is given and used to provide explicit solutions in terms of the original data of the input-output system. Each part concludes with a set of notes describing relevant literature.

The material is nicely organized, and the book is pleasant to read. Many illustrative examples are presented. The text is largely self-contained and requires few prerequisites (linear algebra and some complex function theory will do for most of the text). Part 1 could easily serve as notes for an advanced linear algebra course on the general theory of rational matrix-valued functions. Part 6 and a selection of Parts 1, 4, and 5 provide material for a mathematically oriented graduate course on H-infinity control and interpolation.

There are various other methods to deal with rational matrix function interpolation problems (see, e.g., the recent books Dym [13], Foias-Frazho [14], Helton [23], and Woerdeman [34]). The present book is unique in the fact that it remains entirely in a finite-dimensional context (which seems to be the natural mathematical environment for rational matrix-valued problems) and aims directly at explicit formulas for solutions. In the latter, the book is very effective indeed.

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