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BOOK REVIEW

The cohomology of groups, by Leonard Evens. Clarendon Press, Oxford, 1991, xii+159 pp., \$39.95. ISBN 0-19-853580-5

Let G be a group and consider the functor on G-modules sending M to M^G, the submodule of elements fixed by G. The right derived functors of this are, by definition, the cohomology groups $H^n(G, M)$ of G with coefficients in the G-module M. Since $M^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$, we can also simply define $H^n(G, M)$ to be $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$.

These functors have a long and interesting history. For example, H^1 occurred implicitly in Hilbert's Theorem 90, while H^1 and H^2 found many important applications in the work of Schur, Speiser, Schreier, Brauer, Noether, and others. The general theory, however, owes its origin to work of Hurewicz and Hopf in algebraic topology. An excellent account of the development of this theory is given by Mac Lane [M].

The theory found immediate applications in algebraic topology. Soon after, the use of cohomology was introduced into classfield theory by Artin, Hochschild, Nakayama, and Tate. This application led to a number of excellent texts covering the material needed for this application, such as [L, Ba1] and the relevant chapters of [CE, Se, CF].

Once the cohomology theory of groups was established as an important branch of mathematics, it was natural to consider properties of the theory itself without the need to justify its use by applications. These investigations were mainly concerned with the problem of relating properties of the cohomology $H^*(G, M)$ to group theoretical properties of G. In the case of infinite groups the main concern has been largely with problems of cohomological dimension. We say $cdG \leq n$ if $H^i(G, M) = 0$ for all $i > n$ and all G-modules M. For example, it is known that $cdG \leq 1$ if and only if G is free. Quite a bit of work has been done on finding other classes of groups with $cdG < \infty$. If these groups satisfy appropriate finiteness properties, one can also define Euler characteristics $\chi(G, M)$ with many interesting properties. An excellent source for all of this is Brown's book [Br], which also contains a very nice exposition of the general cohomology theory of groups. See also [G, Ba2, St].

For finite groups there is no problem of cohomological dimension, and recent work on the subject has developed in quite a different direction. Like the cohomology of groups itself, this work has its source in algebraic topology. It is a classical problem to determine which finite groups G can act freely on a sphere S^n. A necessary condition for this is easily found using cohomology theory: the cohomology of G must be periodic, i.e., $H^i(G, M) = H^{i+n+1}(G, M)$ for all $i > 0$ and all modules

M. This condition was studied by Artin and Tate [CE], who showed that a finite group G has periodic cohomology if and only if it contains no subgroup of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Such groups were classified by Zassenhaus and Suzuki. In this way, the problem is reduced to the study of certain explicit groups. This theorem suggested that the rate of growth of the size of $H^n(G, M)$ might be related to the size of elementary abelian subgroups of G, i.e., those of the form $(\mathbb{Z}/p\mathbb{Z})^d$. Define the p-rank of G to be the largest such d. In the early 1960s the following theorem was conjectured by Atiyah, Evens, and the reviewer.

Theorem 1. $\dim_{\mathbb{F}_p} H^n(G, \mathbb{F}_p) \leq Cn^d$ *where* d *is the* p*-rank of* G *and* C *is a sufficiently large constant. This estimate is the best possible in the sense that* $n^{-d} \dim H^n(G, \mathbb{F}_p)$ *does not tend to* 0 *as* n *becomes large.*

This conjecture was then proved by Quillen [Q] using topological methods. A few years later, Quillen and Venkov found a purely algebraic proof of the theorem. These results served as the starting point for much of the subsequent research on the cohomology of finite groups, which is the main focus of Evens's book.

Quillen actually proved a stronger theorem. Let G be a finite group, and let k be a field of characteristic p. Let E run over all elementary abelian p-subgroups of G, and consider the map $H^n(G, k) \to \prod_E H^n(E, k)$. Elementary naturality properties show that the image is contained in the inverse limit of the $H^n(E, k)$ with respect to the maps induced by all $E_1 \to E_2$ of the form $x \mapsto gxg^{-1}$ for some g in G.

Theorem 2 [Q]. *The kernel of* $\varphi \colon H^n(G, k) \to \lim H^n(E, k)$ *is nilpotent, and if* α *is any element of* $\lim H^n(E, k)$*, then there is a power* q *of* p *with* $\alpha^q \in \operatorname{im} \varphi$*.*

Quillen observes that one can give a nice geometric interpretation of this. Since $H^*(G, k)$ is known to be a finitely generated k-algebra by a theorem of Evens and Venkov, it is natural to consider the associated algebraic variety $X_G = \operatorname{Spec} H^*(G, k)$. Note that $H^*(G, k)$ is skew commutative, so one should really take $H^{ev}(G, k) := \bigoplus H^{2n}(G, k)$ if p is odd.

Remark. Since $H^*(G, k)$ is graded, it would be even more natural to consider the associated projective variety $P_G = \operatorname{Proj} H^*(G, k)$. For example, if $H^{d*}(G, k) = \bigoplus H^{dn}(G, k)$, the Veronese embedding shows that $P_G = \operatorname{Proj} H^{d*}(G, k)$. If $G \neq 1$ has periodic cohomology of period d, then $H^{d*}(G, k) = k[x]$, so P_G is a point. However, it is traditional to work with the affine cone X_G.

Well-known results relating $\dim P_G$ to the Hilbert function of its coordinate ring $H^*(G, k)$ show that Theorem 1 is equivalent to the assertion that $\dim X_G = p\text{-rank}\, G$. In other words, the Krull dimension of $H^*(G, k)$ is equal to the p-rank of G. In particular, G has periodic cohomology if and only if $\dim X_G \leq 1$.

One can get a good idea of the structure of X^G by observing that, up to purely inseparable extensions (and, in particular, set theoretically), X_G looks like Spec $\lim H^*(E, k)$ because of Theorem 2. This is obtained from the varieties $\operatorname{Spec} H^*(E, k)$ by making the appropriate identifications. Now $\operatorname{Spec} H^*(E, k)$ is just an affine space of dimension $= p$-rank E. To make the identifications, we first collapse $\operatorname{Spec} H^*(E, k)$ under the action of $N_G(E)/C_G(E)$, the group of automorphisms of E induced by conjugation in G. The resulting pieces then fit together to form Spec $\lim H^*(E, k)$. Expressed more carefully, this construction gives the Quillen stratification of X_G, which gives a very clear picture of the appearance of X_G.

A very important generalization of Quillen's work was introduced by Alperin and Evens. Suppose M is a finitely generated kG-module. Then $H^*(G, M)$ is a finitely generated $H^*(G, k)$-module and so are $\mathrm{Ext}^*_{kG}(M, N)$ and $\mathrm{Ext}^*_{kG}(N, M)$ for any finitely generated kG-module N. Let I_M be the annihilator of either of these Ext functors. Then I_M is a homogeneous ideal of $H^*(G, k)$, and we can consider its zero set $X_G(M) \subset X_G$. It suffices to take $N = M$ by a remark of Carlson. Quillen's results turn out to have analogues in this more general setting (Alperin, Evens, Avrunin, Scott). The dimension of $X_G(M)$ is called the complexity of M, $cx_G(M) = \dim X_G(M)$. As in Theorem 1, we have $cx_G(M) = \max cx_E(M)$ over the elementary abelian p-subgroups E of G. Moreover, $cx_G(M) = 0$ if and only if M is projective, and $cx_G(M) = 1$ if and only if M has a periodic projective resolution (and M is not projective). We also have $X_G(M \otimes N) = X_G(M) \cap X_G(N)$. Using this, Carlson has shown that any Zariski closed subset of P_G can be $P_G(M)$ for some finitely generated module M.

Evens's book gives a very clear and thorough discussion of all of the above material. His methods are algebraic and require nothing beyond standard homological and commutative algebra. In writing a book of this sort, one always has a problem of how much standard material to include. One could simply refer to previous expositions, but this is rather unsatisfactory. Evens found an excellent compromise. His first four chapters (and half of Chapter 5) contain all that is needed from the general theory, but the length is kept to a minimum by leaving many details to the reader. Anyone with a bit of experience in homological algebra should have no difficulty in filling in what is missing and will learn quite a bit in doing so.

The new material begins with Chapter 5. None of this material seems to have appeared before in a textbook. Of particular interest is Chapter 6 on Evens's norm map, a kind of multiplicative transfer which is used very effectively in many places in subsequent chapters. Some other points covered include Serre's theorem characterizing elementary abelian p-groups and Evens's finiteness theorem which generalizes to modules the theorem of Evens and Venkov referred to above.

The remaining chapters then treat in detail the results of Quillen outlined above as well as the theory of the varieties $X_G(M)$ and the complexity of kG-modules for fields k of characteristic p. The discussion is very complete and covers everything one needs to know in order to read the latest papers in the field.

This book should definitely be in the possession of anyone interested in the cohomology theory of groups. Evens has done a great service in making all of this material available in a well-organized and very readable presentation.

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