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BOOK REVIEW

Mathematical analysis and numerical methods for science and technology, Volume 5. Evolution Problems 1, by Robert Dautray and Jacques-Louis Lions, translated from French by Alan Craig. Springer-Verlag, Berlin, 1992, xiv+709 pp., $120.00. ISBN 3-540-50205-X

The book under consideration is the fifth volume of a six-volume work on differential equations. The first four volumes treat stationary problems, i.e., problems that are independent of time. This volume begins the study of evolution problems. A typical example of such problems is the following: find a solution $u(x,t)$ of

$$\frac{du}{dt} + Au = f, \qquad x \in \Omega,\ t < 0, \tag{1}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega \tag{2}$$

with u prescribed on the boundary of the domain Ω and A a differential or integrodifferential operator. Each of the five chapters contained in this volume approaches the solution of such problems from a different perspective. The unifying objective pursued in all the chapters is the mathematical analysis of linear problems that model various physical phenomena. Even though many natural phenomena lead to nonlinear systems of partial differential equations, it is also the case that linearization proves to be an effective tool for the study of these systems. It is therefore useful to be familiar with the linear methods and theories presented in this volume.

An interesting feature of this book is that each chapter is self-contained, beginning with an introduction that gives an overview and orients the reader with its contents and concluding with a review of the material. The reader therefore does not have to read the chapters sequentially in order to follow the methods and analysis developed in each one. The disadvantage of this feature is that the reader who prefers to peruse successive chapters is faced with a repetition of material. Some of the repeated material can be circumvented by a careful examination of the detailed table of contents, which clearly indicates how each chapter is subdivided and where specific material is presented.

Chapter XIV begins Volume 5 of *Mathematical Analysis and Numerical Methods for Science and Technology* with the study of Cauchy problems in $\mathbb{R}^n$ for equations of the form $du/dt + Au = 0$. If $u(0) = u_0$ is given in a function space with finite dimension m, A is a bounded operator (an $m \times m$ matrix) and the solution is given by

$$u(t) = \exp(-At)u_0 \quad \forall t \in \mathbb{R} \tag{3}$$

where the set $\exp(-At)$, $t \in \mathbb{R}$, has the properties of a group. Problems in which the linear operator A is unbounded are also considered in this first chapter; these include Cauchy problems for homogeneous diffusion equations, wave equations, and the Schrödinger equation. In each case, the solution of the problem can be given by the formula

$$u(t) = G(t)u_0 \tag{4}$$

where $G(t)$ is a bounded operator in some function space X, e.g., $\mathcal{L}^2(\mathbb{R}^n)$ that satisfies the properties

$$\begin{aligned} &(1) \quad G(t+s) = G(t)G(s), \ t, s \in \mathbb{R} \text{ or } t, s \geq 0; \\ &(2) \quad G(0) = I \text{ (identity)}; \\ &(3) \quad \lim_{t \to 0} G(t)u_0 = u_0 \text{ in } X\,. \end{aligned} \tag{5}$$

Diffusion, wave, and Schrödinger problems in $\mathbb{R}^n$ can be viewed as particular cases of a more general Cauchy problem for evolution equations related to convolution products. Because such problems are invariant under translations in $\mathbb{R}^n$, they may be analyzed using the Fourier transform over the spatial variable. Chapter XIV includes a discussion of this method and examples of its application to evolution problems.

Chapter XV is concerned with the problems $du/dt + Au = f$ with $u(0) = u_0$ or $d^2u/dt^2 + Au = f$ with $u(0) = u_1$ and $du/dt(0) = u_2$ where A is a selfadjoint operator. A solution u mapping $[0, T)$ into a Hilbert space H is sought in each case using the Fourier method (separation of variables) which requires knowledge of the eigenvalues and eigenfunctions of the operator A. If we can explicitly calculate the spectrum of A, we can obtain the solution to these problems in the form of a series expansion from which properties of the solution can be deduced. A number of examples are presented in this chapter to illustrate the application of the Fourier method in problems where the operator A has a discrete spectrum.

Explicit solutions to certain evolution problems can also be obtained using the method of the Laplace transform with respect to the variable t. This method is developed and applied in Chapter XVI. With the introduction of the Laplace transform, we are now in a position to solve problems of the form (1) where A is a closed operator with domain in a Banach space. The chapter concludes with a number of applications drawn from diverse sources, including hydrodynamics, viscoelasticity, and electromagnetism.

Chapter XVII studies the method of semigroups for differential equations. The first part is devoted to a description of the properties of certain classes of semigroups and the operators that generate them. The second part of the chapter describes how the method can be used to resolve evolution equations. Examples of families of semigroups that lead to approximation (i.e., numerical) methods are examined. The machinery developed in this chapter permits extension of the solution representation (3) to unbounded operators A over a Banach space. We point out that in order for the method of semigroups to be applicable, the operator A must be the infinitesimal generator of a semigroup of class $\mathcal{C}^0$ and must thus satisfy the conditions laid out in the Hille-Yosida Theorem.

There are five categories of semigroups of class $\mathcal{C}^0$ that play an important role in evolution problems for linear differential equations. Included among these are contraction semigroups whose generators are dissipative or conservative. The Laplace operator Δ in $\mathcal{L}^2(\mathbb{R}^n)$ is an example of such a dissipative generator. Compact semigroups are useful in problems with operators whose resolvents are compact. Such problems are illustrated by the Dirichlet problem for a second-order elliptic operator on an open bounded set. Unitary groups are characterized by generators of the form iA where A is a selfadjoint operator. The Cauchy problem for the Schrödinger equation exemplifies the kind of problem to which these groups are applicable. Closely related to unitary groups are isometric groups which are generated by certain conservative operators. Because there exist isometric semigroups that cannot be extended to a unitary group, this type of semigroup can be exploited in the analysis of mathematical models for conservative irreversible systems. Finally, differentiable semigroups, e.g., holomorphic semigroups, which are generated by operators associated with coercive forms, have been found useful in applications due to their smoothing effects on initial data.

The method of semigroups not only allows construction of an explicit solution representation but also yields information about the asymptotic behavior of these solutions as $t \to +\infty$.

The methods of resolution described in Chapters XIV–XVII do not cover equations with time or spatially dependent coefficients nor are they particularly adaptable to numerical evaluation. The last chapter in this volume deals with variational methods and their application to evolution equations in which the operators are asymmetric or time dependent. The first sections of the chapter prepare the functional framework in which the variational methods are to be adopted. These are followed by sections that analyze general first- and second-order linear evolution problems as well as problems involving integrodifferential or delay equations to which the methods of the earlier chapters are not applicable. Scrutiny of variational methods serves as a point of departure for the investigation of nonlinear problems in the last volume of the series.

This book contains a wealth of valuable information on evolution equations that is organized in a consistent and straightforward manner. The authors have attempted to make the book as user-friendly as possible. A detailed table of contents, a comprehensive table of notation, and a uniform system of designating the subdivision of material in each chapter all contribute to helping the reader locate topics of interest. Although the index is not as extensive as I would prefer, the detailed table of contents adequately compensates for this shortcoming. There are numerous instructive examples as well as a broad variety of physical applications that provide concrete illustrations of the abstract methods developed in each chapter.

While I found this volume to be well conceived, I did not find the text very readable. This is in part due to the awkward, somewhat cumbersome translation from the original French. This is unfortunate since the book could be a valuable reference on linear evolution equations if it read more fluently. Substantial parts of the exposition are laborious enough to obscure the substance of the mathematics. Largely because of this weakness, in addition to the book's hefty selling price, I can be only lukewarm about recommending it to my colleagues.

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