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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, July 1993

0273-0979/93 $1.00 + $.25 per page

Topics in varieties of group representations, by Samuel M. Vovsi. London Mathematical Society Lecture Note Series, vol. 163, Cambridge University Press, Cambridge, 1991, xiv+200 pp., £ 19.95, $34.95. ISBN 0-521-42410-0

The title *Topics in varieties of group representations* consists essentially of four nouns. Of these, the first needs no explanation, varieties will be treated in the next paragraph, and the scope of the last two is as follows. The groups in question are just groups—not Lie groups, not algebraic groups, not formal groups, not quantum groups. The representations are linear over commutative rings. For the purposes of this book a commutative ring K (with unit element) is preassigned, and a group representation is a pair (V, G), where G is an abstract group and V is a KG-module. It is important that both V and G may vary. A morphism $(V_1, G_1) \to (V_2, G_2)$ is a pair (ϕ, ψ), where ϕ is a K-module homomorphism $V_1 \to V_2$, ψ is a group homomorphism $G_1 \to G_2$, and $(vg)\phi = (v\phi)(g\psi)$ for all $v \in V_1$, $g \in G_1$.

Varieties were introduced about sixty years ago by Garrett Birkhoff. They are equationally definable classes of algebraic systems. That is to say, the axioms should be in the form of universally quantified equations like the associative law $x(yz) = (xy)z$ or the commutative law $xy = yx$. A typical example is the class of all groups, for although some descriptions involve existential quantifiers—as in $(\exists x_0)((\forall x)(xx_0 = x_0x = x) \,\&\, (\forall x)(\exists y)(xy = yx = x_0))$—these can be easily edited out of the theory. If the identity element is included as a constant 1 and inverses are produced by a unary operator $^{-1}$, then the axioms can be written as "identical relations" or "laws"

$$(\forall x, y, z)((xy)z = x(yz)), \qquad (\forall x)(x1 = 1x = x), \qquad (\forall x)(xx^{-1} = x^{-1}x = 1).$$

Similarly, the class of rings, the class of commutative rings with 1, the class of associative algebras over a given field, the class of Lie algebras over a given field, and indeed very many of the classes of algebraic systems that occur naturally in mathematics are varieties. There is one instructive example that is not. Most descriptions of the concept of field include the conditions that $0 \neq 1$ and that if $x \neq 0$ then x has a multiplicative inverse. It is conceivable that the field axioms might be rewritten in some ingenious way and presented as universally quantified equations. To see that this is not possible, observe that any variety of algebraic systems has the property that it is closed under the operations known as

S, Q, and C. That is, if $\mathfrak{V}$ is a variety of algebraic systems, then any subalgebra, any epimorphic image, and any cartesian product of algebras in $\mathfrak{V}$ is itself in $\mathfrak{V}$. The class of fields is not closed under these operations, and therefore it is not a variety. Since, for example, the cartesian product (i.e., the direct sum) of two fields has zero-divisors and therefore is not a field, we know that the axioms for fields cannot be massaged into an equational form.

The fact that a variety is closed under S, Q, and C has a converse. Birkhoff's main theorem tells us that any class of algebraic systems closed under the operations S, Q, and C is a variety. As a consequence, the variety generated by an algebra A may be described in two very different ways. On the one hand, it is the class of algebras satisfying all those relations that hold identically in A (these are known as the "laws" or the "identical relations" of A). On the other hand, it is the class of all algebras obtainable from A by the operations S, Q, and C. One might expect that these operations would have to be performed many times, but it can be proved that in fact $\mathrm{VAR}A = \mathrm{Q}(\mathrm{S}(\mathrm{C}\{A\}))$.

The study of varieties of groups was instigated by B. H. Neumann in his Ph.D. thesis of 1935. It is no coincidence that this was about the same time as Birkhoff produced his ideas about varieties of general algebraic systems: they were both at Cambridge, in the intellectual environment of Philip Hall. In the context of identical relations, groups have received more intense attention than has any other type of algebraic system, and the theory of varieties has grown into a rich and satisfying part of group theory. It has had notable successes and still has high ambitions. A question of great interest—the one on which (for expository purposes only) this review will concentrate—is whether or not the laws of a given group, or a given variety, are finitely based. The question is, given a group G or a variety $\mathfrak{V}$, is there a finite set of laws from which all others follow?

Since there are only countably many finite sets of laws, there are only countably many varieties whose laws are finitely based. One of the difficult theorems in the subject, proved by several mathematicians (Ol'shanskii, Vaughan-Lee, Adjan) independently in 1969–70, is that there are uncountably many, in fact, $2^{\aleph_0}$, different varieties of groups. Therefore, in quite a strong sense most varieties are not finitely based. This gives force to the many known theorems and provides the context for a number of open questions.

Theorem 1 [Roger Lyndon, 1952]. *The laws of a nilpotent group are finitely based.*

Problem 1. *Is it true that the laws of a polycyclic group are finitely based?*

Although polycyclic groups are in many ways not much more complicated than finitely generated nilpotent groups, there appears to be a very wide gulf between Theorem 1 and Problem 1.

Theorem 2 [Sheila Oates and Martin Powell, 1964]. *The laws of a finite group are finitely based.*

Problem 2. *Is it true that if H is a subgroup of finite index in a group G, then the laws of G are finitely based if and only if the laws of H are finitely based?*

A group is said to be $\mathscr{X}$-by-$\mathscr{Y}$ if it has a normal subgroup with property $\mathscr{X}$, whose factor group has property $\mathscr{Y}$. In one direction Problem 2 is essentially the question whether a (finitely based)-by-finite group is finitely based. An important special case is

Problem 2(a). *Is it true that the laws of a nilpotent-by-finite group are finitely based?*

It is not even known yet (at least, not to the reviewer) whether the laws of an abelian-by-finite group are finitely based.

Theorem 3 [A. N. Krasil′nikov, 1990]. *The laws of a nilpotent-by-abelian group are finitely based.*

Problem 3. *Is it true that the laws of an abelian-by-nilpotent group are finitely based?*

Among the most important varieties of groups are the so-called Burnside varieties. The Burnside variety of exponent n is the variety $\mathfrak{B}_n$ defined by the law $x^n = 1$. If we speak of a variety of exponent n (or of exponent dividing n), we mean a variety in which every group has exponent dividing n, that is, a subvariety of $\mathfrak{B}_n$. Many of the known non-finitely-based varieties have finite exponent. Indeed, there are varieties of exponent 8 that are not finitely based.

Theorem 4 [Michael Atkinson, 1973]. *Every variety of exponent* 6 *is finitely based.*

Problem 4. *Is it true that every variety of exponent* 4 *is finitely based?*

In each case an enormous amount is known about the groups (or the varieties) referred to in the problem. After some work of Philip Hall, for example, there is quite a strong sense in which abelian-by-nilpotent groups form a substantially better understood class than nilpotent-by-abelian groups. Nevertheless, Krasil′nikov's theorem, which essentially completes a long line of research begun by D. E. Cohen in 1965, has not yet been matched by any real progress on Problem 3.

The conventional algebraic systems to which Birkhoff's theory applies consist of a set with certain preassigned operations (multiplication, inverse, and unit element in the case of groups; addition, additive inverse, 0, and, for each element a of K, the unary operation of 'scalar' multiplication by a, in the case of modules over the ring K). Since group representations are 'two-sorted', they do not quite fit this pattern. Nevertheless, the language and the grammar of the theory, as presented in the preliminary chapter of the book under review, are very similar to what is familiar from the theories of varieties of groups and varieties of linear algebras (or PI-rings, i.e., rings with polynomial identities). Indeed, those are its natural parents, under whose strong influence the theory is being developed.

Of these parents, the dominant partner is the theory of varieties of groups, which is why so much of this essay has been devoted to that subject. Much of the book is devoted to research directly inspired by, and often heavily dependent upon, what is known about groups. Chapter 1 treats what are called stable varieties. A representation (V, G) is defined to be stable if there exists a sequence $\{0\} = V_0 \leqslant V_1 \leqslant \cdots \leqslant V_n = V$ of G-submodules such that for each relevant i the action of G on the section V_i/V_{i-1} is trivial. A variety is defined to be stable if it consists entirely of stable representations (in which case there will be a uniform bound to the minimum length n of such sequences for representations (V, G) in the variety). The behaviour of stable varieties of group representations turns out, perhaps not surprisingly, to be essentially no more than an amalgamation of the theories of nilpotent varieties of groups, nilpotent varieties of associative algebras, and nilpotent varieties of Lie algebras. Similarly, much of the material in Chapters 2 and 3 is influenced by the ideas of Graham Higman, Dennis Cross, Sheila Oates, and Martin Powell, which culminated in Theorem 2 and which were substantially developed by L. G. Kovács, M. F. Newman, and others to produce a deep understanding of how the so-called 'critical' groups in a locally finite variety behave.

Although the apron strings have not yet been untied and the subject is not yet earning its keep, there are some signs that it may perhaps have the potential to contribute to the family fortune. Two of the "further topics" treated in the fourth (and last) chapter point in that direction. One of Ol′shanskii's 1971 theorems, classifying certain special varieties of soluble groups, may be formulated in terms of group representations. Likewise, the line of research that was initiated by D. E. Cohen with his 1965 proof that metabelian (i.e., abelian-by-abelian) varieties are finitely based and which was developed by Susan McKay, Michael Vaughan-Lee, Roger Bryant, M. F. Newman, and others until it led to the splendid result of Krasil′nikov stated as Theorem 3 above has a close affinity with the theory of varieties of group representations. One of the main theorems of the book states that if the coefficient ring K is a field and if (V, G) is stable-by-finite (that is, G has a subgroup of finite index whose action on V is stable in the sense described above), then the laws of (V, G) have a finite basis. If this theorem, which was published in 1987 by Nguyen Hung Shon and the author, could be proved without the restriction that K be a field, it would almost certainly lead to a positive solution of Problem 2(a). Since a polycyclic group is nilpotent-by-abelian-by-finite, it might then be possible to amalgamate this line of thinking with Krasil′nikov's ideas so as to produce also a positive solution to Problem 1.

The book presents to non-Russian readers a theory that has been developed almost entirely in the former Soviet Union. It is clear and hardly suffers at all from the fact that English is not the author's first language. Nevertheless, it is not particularly well written. For example, the author allows himself unnecessary or unattractive neologisms such as "3-tuple" and "comonolithic". Sometimes he attaches the wrong symbol to quantifiers, as in "for every $1 \neq g \in G$". Such usages are not excused by the sad fact that they are becoming common even amongst writers whose first language is English (or American). Blemishes like these are irritating, but they do not interfere badly between author and reader. The fact is that the book is well organized and easy to understand.

One test of the success of a monograph is whether it fuels its own obsolescence. The 1967 book *Varieties of groups* by Hanna Neumann inspired a great deal of work that led to a number of major advances in its subject, and a completely new edition has been needed for some time. Dr. Vovsi's book appears to have been inspired by, and certainly is in some ways similar to, that older work. We wish it the same autodestructive success.

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