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QUASIPOSITIVITY AS AN OBSTRUCTION TO SLICENESS

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Abstract. For an oriented link $L \subset S^3 = \partial D^4$, let $\chi_s(L)$ be the greatest Euler characteristic $\chi(F)$ of an oriented 2-manifold F (without closed components) smoothly embedded in D^4 with boundary L. A knot K is *slice* if $\chi_s(K) = 1$. Realize D^4 in $\mathbb{C}^2$ as $\{(z,w) : |z|^2 + |w|^2 \leq 1\}$. It has been conjectured that, if V is a nonsingular complex plane curve transverse to S^3, then $\chi_s(V \cap S^3) = \chi(V \cap D^4)$. Kronheimer and Mrowka have proved this conjecture in the case that $V \cap D^4$ is the Milnor fiber of a singularity. I explain how this seemingly special case implies both the general case and the "slice-Bennequin inequality" for braids. As applications, I show that various knots are not slice (e.g., pretzel knots like $\mathcal{P}(-3,5,7)$; all knots obtained from a positive trefoil $O\{2,3\}$ by iterated untwisted positive doubling). As a sidelight, I give an optimal counterexample to the "topologically locally-flat Thom conjecture".

1. A brief history of sliceness

A *link* is a compact 1-manifold without boundary L (i.e., finite union of simple closed curves) smoothly embedded in the 3-sphere S^3; a *knot* is a link with one component. If S^3 is realized in $\mathbb{R}^4$ as, say, the unit sphere, then a natural way to construct links is to intersect suitable two-dimensional subsets $X \subset \mathbb{R}^4$ with S^3; one may then ask how constraints on X are reflected in constraints on the link $X \cap S^3$.

For instance, Fox and Milnor (c. 1960) considered, in effect, the case that X is a smooth 2-sphere intersecting S^3 transversally; at Moise's suggestion, Fox [5] adopted the adjective *slice* to describe the knots and links $X \cap S^3$ so constructed. Fox and Milnor [6] gave a criterion for a knot K to be slice: its *Alexander polynomial* $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ must have the form $F(t)F(t^{-1})$. This shows that, for instance, the two trefoil knots $O\{2, \pm 3\}$ are not slice (since $\Delta_{O\{2,\pm 3\}} = t^{-1} - 1 + t$ is not of the form $F(t)F(t^{-1})$), but it says nothing about the two granny knots $O\{2,3\} \# O\{2,3\}$, $O\{2,-3\} \# O\{2,-3\}$ (indeed, both granny knots share the Alexander polynomial

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$(t^{-1}-1+t)^2$ with the square knot $O\{2,3\} \# O\{2,-3\}$, which *is* slice), and Fox could only aver [5] that "it is highly improbable that the granny knot is a slice knot."

By the end of the 1960s, several mathematicians [30, 14, 31] had found invariants which could be applied to show that, for instance, the granny knots are not slice. For any knot K, all these invariants (signatures of various families of hermitian forms), as well as $\Delta_K(t)$, can be calculated from the *Seifert pairing* $\theta_F : H_1(F,\mathbb{Z}) \times H_1(F,\mathbb{Z}) \to \mathbb{Z}$ determined by any *Seifert surface* F for K (i.e., a smooth, oriented, 2-submanifold-with-boundary $F \subset S^3$ without closed components, with $K = \partial F$). In particular, if K is slice, then (for any F) there is a subgroup $N \subset H_1(F,\mathbb{Z})$ with $\operatorname{rank}(N) = \frac{1}{2}\operatorname{rank}(H_1(F,\mathbb{Z}))$ on which θ_F vanishes identically. Any knot for which such a subgroup exists is called *algebraically slice* (briefly, A-slice). Levine showed [13] that in higher odd dimensions, A-slice knots are slice. Whether this were true for knots in S^3 was unknown until 1975, when Casson and Gordon [1, 2] developed "second-order" obstructions to sliceness (again using signatures, but of more subtly constructed forms that are not determined just by a Seifert pairing) and used them to show that many A-slice knots are not slice. Their methods were powerless, however, to prove nonsliceness of any knot K with $\Delta_K(t) = 1$ (such a knot is A-slice, as observed by L. Taylor, cf. [9 (1978), problem 1.36]).

The subject took surprising turns in the 1980s after Donaldson and Freedman revolutionized the theory of 4-manifolds and, not so incidentally, the theory of knots and links in S^3. In fact, let $X \subset \mathbb{R}^4$ now be a 2-sphere, still transverse to S^3, which is, however, assumed no longer smooth but merely *topologically locally-flat* (i.e., in local $\mathcal{C}^0$ charts it looks like $\mathbb{R}^2 \subset \mathbb{R}^4$); then the link $X \cap S^3$ is *topologically locally-flatly slice* (briefly, T-slice). T-slice implies A-slice. Freedman [8] proved that any knot K with $\Delta_K(t) = 1$ (e.g., any *untwisted double*) is T-slice. Nonsliceness results flowed from Donaldson's restrictions on intersection forms of smooth, as distinct from topological, 4-manifolds: Casson proved the existence of a nonslice knot K with $\Delta_K(t) = 1$ (cf. [9 (1984), problem 1.36]); Akbulut gave an explicit example of such a knot, the untwisted positive double $D(O\{2,3\}, 0, +)$ (cf. [3]); Cochran and Gompf [3] found large classes of knots K such that $D(K, 0, +)$ is not slice; and Yu [32], building on work of Fintushel and Stern, found many A-slice Montesinos knots which are not slice.

In §4 I give many examples of nonslice knots: for example—recovering some of Fintushel and Stern's results—all pretzel knots $\mathcal{P}(p,q,r) \neq O$ with Alexander polynomial 1, and—considerably generalizing [3, Corollary 3.2]—all iterated untwisted positive doubles of any knot $K \neq O$ which is a closed positive braid. The method in each case is to show that the knot under consideration is *strongly quasipositive*, then to use the fact that a strongly quasipositive knot $K \neq O$ is not slice, which follows from a corollary to a recent result of Kronheimer and Mrowka [10]. In §3 I state their result and establish that corollary, as well as a superficially stronger (actually equivalent) corollary, the "slice-Bennequin inequality" for braids. Section 2 is preliminary material on quasipositivity, etc. Section 5 is a sidelight, using an example from §2 to produce a topologically locally-flat surface in $\mathbb{CP}^2$, of algebraic and geometric degree 5, with genus $5 = \frac{1}{2}(5-1)\times(5-2)-1$: this is an optimal counterexample to the "topologically locally-flat Thom conjecture".

Remarks. (1) Note that it is not Kronheimer and Mrowka's machinery, but "only" their (spectacular) result which is used. In particular, one can understand the present note while staying totally disengaged from gauge theory.

(2) Although Kronheimer and Mrowka in [10] do not discuss the slice- Bennequin inequality, they do draw explicit attention to a (strictly weaker) corollary of their main result, namely, the affirmative answer to the "question of Milnor" [15] on the unknotting number of a link of a singularity. The nonsliceness results of the present paper have nothing to do with unknotting number.

(3) W. M. Menasco has recently announced a proof of the unknotting result which, in marked contrast to that in [10], uses purely three-dimensional techniques (somewhat in the style of [0]); should such techniques someday be used successfully to establish the slice-Bennequin inequality, then the present nonsliceness results will have a purely three-dimensional proof as well.

2. Quasipositivity

Transverse $\mathbb{C}$-links and quasipositive Seifert surfaces. When constructing links as intersections $X \cap S^3$, instead of restricting the topological type of X as in §1, one might restrict the nature of the embedding $X \hookrightarrow \mathbb{R}^4$. In particular, if $\mathbb{R}^4$ is identified with $\mathbb{C}^2 \supset S^3 := \{(z,w) : |z|^2 + |w|^2 = 1\}$ and X is required to be a complex plane curve, then one can obtain many interesting links.

Definitions. A *complex plane curve* is any set $V_f := f^{-1}(0) \subset \mathbb{C}^2$, where $f(z,w) \in \mathbb{C}[z,w]$ is nonconstant; V_f is a smooth, oriented 2-submanifold of $\mathbb{C}^2$ except at a finite set $\mathcal{S}(V_f) \subset V_f$ of singularities. If V_f is transverse to S^3, then the oriented link $K_f := V_f \cap S^3$ is a *transverse* $\mathbb{C}$*-link* [22, 29].

Examples. Replacing S^3 by a round sphere of sufficiently small radius centered at a point of $\mathcal{S}(V_f)$, one sees that any *link of a singularity* of a complex plane curve is a transverse $\mathbb{C}$-link; replacing S^3 by a round sphere of sufficiently large radius, one sees that any *link at infinity* of a complex plane curve is a transverse $\mathbb{C}$-link.

Links of singularities and links at infinity, though very interesting (cf. [15, 11, 4, 23, 17], etc.), are highly atypical transverse $\mathbb{C}$-links (for instance, while the unknot O is the only slice knot which is a link of a singularity [11] or a link at infinity [23], many nontrivial slice knots are transverse $\mathbb{C}$-links [19]). A much broader class of transverse $\mathbb{C}$-links is easily defined using braid theory.

Definitions. In the *n-string braid group*

$$B_n := \mathrm{gp}\left(\sigma_i, 1 \le i \le n-1 \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = \sigma_j^{-1}\sigma_i, & |i-j| = 1 \\ [\sigma_i, \sigma_j] = 1, & |i-j| \ne 1 \end{array}\right),$$

a *positive band* is any conjugate $w\sigma_i w^{-1}$ ($w \in B_n, 1 \le i \le n-1$); a *positive embedded band* is one of the positive bands $\sigma_{i,j} := (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1}$, $1 \le i < j \le n$ (e.g., each *standard generator* $\sigma_i = \sigma_{i,i+1}$ is a positive embedded band). A (*strongly*) *quasipositive braid* is any product of positive (embedded) bands (e.g., a *positive braid*, that is, a product of standard generators, is strongly quasipositive). A (*strongly*) *quasipositive* oriented link is one which can be realized as the closure of a (strongly) quasipositive braid. Up to ambient isotopy, every quasipositive link is a transverse $\mathbb{C}$-link [19].

Question. Is every transverse $\mathbb{C}$-link quasipositive?

Remarks. (1) There are non-quasipositive knots, for example, the figure-8. This follows, for instance, from a result of Morton [16] and Franks and Williams [7] about the *oriented link polynomial* of a closed braid (cf. [26]). (Note, however, that every Alexander polynomial, and indeed every Seifert pairing, can be realized by a quasipositive knot or link [21].)

(2) There are knots which are not transverse $\mathbb{C}$-links; the figure-8 is again an example. Biding an affirmative answer to the above question, I know of no way to show this without using the methods of the present paper.

Any specific expression of a quasipositive braid as a product of positive bands, $\beta = w_1\sigma_{i_1}w_1^{-1}w_2\sigma_{i_2}w_2^{-1}\cdots w_k\sigma_{i_k}w_k^{-1} \in B_n$, gives a recipe for constructing a *quasipositive braided Seifert ribbon* $S(w_1\sigma_{i_1}w_1^{-1},\dots,w_k\sigma_{i_k}w_k^{-1}) \subset D^4$, that is, a smooth surface (actually "ribbon-embedded", a refinement we can ignore) bounded by the closed braid $\widehat{\beta}$. The isotopy carrying $\widehat{\beta}$ onto a transverse $\mathbb{C}$-link K_f can be chosen to carry $S(w_1\sigma_{i_1}w_1^{-1},\dots,w_k\sigma_{i_k}w_k^{-1})$ onto the (nonsingular) piece of complex plane curve $V_f \cap D^4$. The Euler characteristic of $S(w_1\sigma_{i_1}w_1^{-1},\dots,w_k\sigma_{i_k}w_k^{-1})$ is $n-k$. If $\beta = \sigma_{i_1,j_1}\sigma_{i_2,j_2}\cdots\sigma_{i_k,j_k}$ is strongly quasipositive, then $S(\sigma_{i_1,j_1},\sigma_{i_2,j_2},\dots,\sigma_{i_k,j_k}) \subset D^4$ is the "push-in" of a *quasipositive braided Seifert surface*, abusively indicated by the same notation; Figs. 1 and 2 give a sufficient idea of the construction.

A Seifert surface is *quasipositive* if it is ambient isotopic to a quasipositive braided Seifert surface. (See [20–22] for more on braided surfaces and quasipositivity.) A subset of a surface is *full* if no component of its complement is contractible.

Theorem [18]. *A full subsurface of a quasipositive Seifert surface is quasipositive.*

Plumbing; quasipositive doubles. For K a knot, $\tau \in \mathbb{Z}$, let $A(K,\tau) \subset S^3$ be an *annulus of type K with τ twists*; that is, $K \subset \partial A(K,\tau)$ and $\theta_{A(K,\tau)}$ has matrix (τ). Let $A(K,\tau)*A(O,\pm1)$ be a Seifert surface formed by *plumbing* $A(O,\pm1)$ to $A(K,\tau)$; that is, there is a 3-cell $B \subset S^3$ such that $A(K,\tau) \subset B$, $A(O,\pm1) \subset S^3 \setminus \operatorname{Int} B$, and $A(K,\tau) \cap A(O,\pm1) = A(K,\tau) \cap \partial B = A(O,\pm1) \cap \partial B$ is a quadrilateral 2-cell whose sides are, in order, contained in alternate components of $\partial A(K,\tau)$ and $\partial A(O,\pm1)$. The knot $D(K,\tau,\pm) := \partial(A(K,\tau) * A(O,\mp1))$ is the *τ-twisted positive* (resp. *negative*) double of K. A matrix for $\theta_{D(K,\tau,\pm)}$ is $\left(\begin{smallmatrix}\tau & 1\\ 0 & \mp1\end{smallmatrix}\right)$, so $\Delta_{D(K,\tau,\pm)}(t) = 1 \mp \tau(t-2+t^{-1})$, and $D(K,0,\pm)$ is A-slice for any K.

Lemma 1. *If $K \neq O$ is strongly quasipositive, then $A(K,0)$ is quasipositive.*

Proof. This follows from the last theorem; for a collar of the boundary of a quasipositive Seifert surface $F \neq D^2$ bounded by K is an annulus $A(K,0)$, and full. □

Example. $O\{2,3\} = \partial S(\sigma_1,\sigma_1,\sigma_1)$; $A(O\{2,3\},0)$ is isotopic to the quasipositive braided surface $S(\sigma_{3,6},\sigma_{1,4},\sigma_{3,5},\sigma_{4,6},\sigma_{2,5},\sigma_1)$ pictured in Fig. 1.

Lemma 2. *If the knot $K \neq O$ is strongly quasipositive, then $D(K,0,+)$ is strongly quasipositive, being the boundary of a quasipositive braided Seifert surface of Euler characteristic -1.*

Proof. This follows from Lemma 1 and a theorem in [25]: for any Seifert surface S, annulus A, and proper arc $\alpha \subset S$, the plumbed surface $S *_\alpha A$ is quasipositive if both S and A are quasipositive. A proof in the present case, where S is itself an annulus, α is a transverse arc of S, and $A = A(O,-1)$, was given in [22]; the reader can readily recreate it after comparing the following example to the preceding one. □

Example. $D(O\{2,3\},0,+) = \partial S(\sigma_6,\sigma_{3,6},\sigma_6,\sigma_{1,4},\sigma_{3,5},\sigma_{4,6},\sigma_{2,5},\sigma_1)$.

FIG. 1 $S(\sigma_{3,6}, \sigma_{1,4}, \sigma_{3,5}, \sigma_{4,6}, \sigma_{2,5}, \sigma_1)$.

FIG. 2 $F(-3,5,7)$ on the Seifert surface of $O\{5,5\}$.

Quasipositive pretzels. Let $p, q, r \in \mathbb{Z}$. A diagram for the *pretzel* link $\mathcal{P}(p,q,r)$ is obtained from a braid diagram for $\beta_{p,q,r} := \sigma_1^{-p}\sigma_3^{-q}\sigma_5^{-r} \in B_6$ by forming the *plat* of $\beta_{p,q,r}$ (using the pairing (16)(23)(45) at top and bottom). If p, q, r are all odd, then $\mathcal{P}(p,q,r)$ is a knot, and (once it is oriented) the obvious surface $F(p,q,r)$ that it bounds (two 0-handles attached by three 1-handles) is a Seifert surface.

Example. $\mathcal{P}(1,1,1) = O\{2,3\}$; $F(1,1,1) = S(\sigma_1, \sigma_1, \sigma_1)$ (up to ambient isotopy).

Lemma 3. *For p, q, r all odd, $F(p,q,r)$ is quasipositive iff*

$$(*) \qquad \min\{p+q, p+r, q+r\} > 0.$$

Proof. For $-\tau \in \{p+q, p+r, q+r\}$, $F(p,q,r)$ contains $A(0,\tau)$ as a full subsurface (omit each 1-handle in turn). It is proved in [29] that $A(O,\tau)$ is quasipositive iff $\tau < 0$; therefore, by the theorem of [18] quoted above, if $F(p,q,r)$ is quasipositive, then $(*)$ is true. Conversely, if $(*)$ is true, then either $\min\{p,q,r\} > 0$, or exactly one of p, q, r is negative and it is of strictly smaller absolute value than the other two. In the first case, $F(p,q,r)$ is obtained (up to ambient isotopy) from the quasipositive Seifert surface $S(\sigma_1, \sigma_1, \sigma_1)$ by applying nonpositive twists to the three 1-handles, so, according to [21] (or [22]), $F(p,q,r)$ is quasipositive; a similar, only slightly less straightforward, twisting argument applies in the second case. □

Example. $F(-3,5,7)$ is ambient isotopic to

$$S(\sigma_1, \sigma_2, \sigma_{2,4}, \sigma_{3,6}, \sigma_{1,4}, \sigma_5, \sigma_{2,5}).$$

3. Kronheimer-Mrowka Theorem; "slice-Bennequin inequality"

If $L \subset S^3$ is an oriented link, let $\chi_s(L)$ be the greatest Euler characteristic $\chi(F)$ of an oriented 2-manifold F (without closed components) smoothly embedded in D^4 with boundary L; so, for a knot K, $\chi_s(K) = 1$ iff K is slice.

If $K_f \subset S^3_\epsilon$ is the link of the singularity $(0,0) \in \mathcal{S}(V_f)$, then its *Milnor fiber* [15] is the nonsingular piece of complex plane curve $V_{f-\delta} \cap D^4_\epsilon$ (for any sufficiently small $\delta > 0$); of course, K_f is isotopic to $K_{f-\delta} = \partial V_{f-\delta} \cap D^4_\epsilon$. The following is a restatement of [10, Corollary 1.3] in the present terminology.

Kronheimer-Mrowka Theorem. *If K_f is the link of a singularity, then $\chi_s(K_f)$ is the Euler characteristic of its Milnor fiber.*

This is a special case of the next proposition, which, however, it implies!

Proposition. *If $K_f \subset S^3$ is a transverse $\mathbb{C}$-link and $\mathcal{S}(V_f) \cap D^4 = \emptyset$, then $\chi_s(K_f) = \chi(V_f \cap D^4)$.*

Proof. Without loss of generality (after perturbing f slightly) we may assume that the projective completion $\Gamma \subset \mathbb{CP}^2$ of V_f in $\mathbb{CP}^2 \supset \mathbb{C}^2$ is nonsingular and transverse to the line at infinity. Then the link at infinity of V_f is isotopic to $O\{d,d\}$, $d = \deg \Gamma$. Assuming $\chi_s(K_f) > \chi(V_f \cap D^4)$, we would then also have $\chi_s(O\{d,d\}) > \chi(V_f)$. Yet $O\{d,d\}$ is also a link of a singularity (namely, $z^d + w^d$ at the origin), and the interior of its Milnor fiber is diffeomorphic to V_f, so our assumption is inconsistent with the Kronheimer-Mrowka Theorem. □

Corollary. *If $\beta = w_1 \sigma_{i_1} w_1^{-1} \cdots w_k \sigma_{i_k} w_k^{-1} \in B_n$ is quasipositive, then $\chi_s(\widehat{\beta}) = n - k$.* □

This corollary—in fact, its special case that a strongly quasipositive knot $K \neq O$ is not slice—suffices for the applications in §4. It is easy, however, to go further. Let $e : B_n \to \mathbb{Z}$ be abelianization (exponent sum with respect to the standard generators σ_i).

Slice-Bennequin Inequality. *For every n, for every $\beta \in B_n$, $\chi_s(\widehat{\beta}) \leq n - e(\beta)$.*

Proof. The preceding corollary asserts the slice-Bennequin inequality (with equality) for β quasipositive. Now apply the following lemma. □

Lemma 4 [28]**.** *If the slice-Bennequin inequality holds for all quasipositive β, then it holds for all β.*

Proof. Since [28] is somewhat obscure, I resuscitate the proof. Let

$$\beta = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k} \in B_n, \qquad \epsilon_j \in \{1, -1\},$$

have p (resp. $\nu = k - p$) indices j with $\epsilon_j = 1$ (resp. $\epsilon_j = -1$); so $e(\beta) = p - \nu$. If $1 \leq j_1 < j_2 < \cdots < j_p \leq k$ are the positive indices, let $\gamma = \sigma_{i_{j_1}} \cdots \sigma_{i_{j_p}}$; so γ is quasipositive (in fact, positive), and $\chi_s(\widehat{\gamma}) = n - p$. There is a smoothly embedded surface $Q \subset S^3 \times [0,1]$ of Euler characteristic $-\nu$ (Q is a union of annuli with ν extra 1-handles attached somehow) such that $\partial Q \cap S^3 \times \{0\} = \widehat{\gamma}$ and $\partial Q \cap S^3 \times \{1\} = \widehat{\beta}$; so $|\chi_s(\widehat{\beta}) - \chi_s(\widehat{\gamma})| \leq \nu$, and $\chi_s(\widehat{\beta}) \leq n - p + \nu = n - e(\beta)$. □

Remark. Bennequin [0] proved that $\chi(S) \leq n - e(\beta)$ for all $\beta \in B_n$ and all Seifert surfaces S bounded by $\widehat{\beta}$, and conjectured the slice-Bennequin inequality.

4. Nonsliceness results

Proposition. *If the knot $K \neq O$ is strongly quasipositive, then none of the knots $D^1(K) := D(K, 0, +)$, $D^i(K) := D(D^{i-1}(K), 0, +)$, $i \geq 2$, is slice.*

Proof. If $K \neq O$ is strongly quasipositive, then, by Lemma 2 and the corollary to the Kronheimer-Mrowka Theorem, $D(K, 0, +)$ is strongly quasipositive and not slice (because $\chi_s(D(K, 0, +)) = -1$); the proof is completed by induction. □

Remark. Cochran and Gompf [3, Corollary 3.2] show that if the knot $K \neq O$ is the closure of a positive braid, then $D^i(K)$ is not slice for $1 \leq i \leq 6$. The present result is infinitely stronger. It would be interesting to understand the relation between being (strongly) quasipositive and "being greater than or equal to $\mathbb{T}$" in the sense of [3].

Proposition. *If p, q, r are all odd, $\{1, -1\} \not\subset \{p, q, r\}$, and*

$$(**) \qquad qr + rp + pq = -1,$$

then $\mathcal{P}(p, q, r)$ is not slice.

Remarks. (1) For p, q, r odd, $\{1, -1\} \subset \{p, q, r\}$ iff $\mathcal{P}(p, q, r)$ is an unknot, and $(**)$ iff $\Delta_{\mathcal{P}(p,q,r)}(t) = 1$.

(2) This corollary, which answers problem 1.37 in [9], is a special case of results in [32].

Proof. Not all of p, q, r have the same sign; we may assume $p < 0 < q \leq r$. By Lemma 3, if $(*)$ is true, then $\mathcal{P}(p, q, r)$ bounds a quasipositive Seifert surface of Euler characteristic -1, so by §3 it is not slice. Suppose $(*)$ is false; then $p+q = -a, r-q = b$ with $a, b \geq 0$, so by $(**)$, $-1 = qr+rp+pq = -(q^2+2aq+ab)$, whence $q = 1, a = 0, p = -1$, and $\{1, -1\} \subset \{p, q, r\}$, contrary to hypothesis. □

5. The "topologically locally-flat Thom conjecture"

The "Thom conjecture" says that $(|d_a(S)| - 1)(|d_a(S)| - 2)/2 \leq g(S)$ for any closed, oriented surface S smoothly embedded in $\mathbb{CP}^2$ of (algebraic) degree $d_a(S)$ and genus $g(S)$. This conjecture is not known to be true, but it certainly becomes false if it is strengthened by replacing "smoothly embedded" with "topologically locally-flatly embedded" (briefly, T-embedded). Let the *geometric degree* $d_g(S)$ of a T-embedded surface $S \subset \mathbb{CP}^2$ be the minimum number of points of intersection of a surface S' isotopic to S that intersects $\mathbb{CP}^1_\infty$ transversally.

Claim. There is a T-embedded surface $S \subset \mathbb{CP}^2$ with $g(S) = d_a(S) = d_g(S) = 5$.

Remark. Lee and Wilczyński [12] show the existence, for every $d > 0$, of a T-embedded surface $W_d \subset \mathbb{CP}^2$ with $d_a(W_d) = d$ and $g(W_d) = g_t(d)$, where $g_t(d)$ is the lower bound for $g(S)$ provided by classical estimates (Hsiang and Szczarba, Rohlin, etc.) if $S \subset \mathbb{CP}^2$ is T-embedded and $d_a(S) = d$; $g_t(d) = (d-1)(d-2)/2$ for $1 \leq d \leq 4$, and $g_t(5) = 5$, so the claim is a sharp counterexample. The techniques of [12] appear to give no control over $d_g(W_d)$. It would be interesting to know if W_d can always be taken to have geometric degree d.

Proof (sketch). Follow [27]; instead of replacing a copy of $A(O\{2, 3\}, 0) * A(O, -1)$ embedded on the quasipositive Seifert surface of $O\{6, 6\}$ by a T-embedded disk with the same boundary, do the same with the copy of $F(-3, 5, 7)$ embedded on the quasipositive Seifert surface of $O\{5, 5\}$ illustrated in Fig. 2. (By an oversight, in [27] the embedding actually given was of $A(O\{2, 3\}, 1) * A(O, -1)$.) □

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