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BOOK REVIEW

Geometry of reflecting rays and inverse problems, by V. M. Petkov and L. N. Stoyanov. Wiley, New York, 1992, vi+313 pp. ISBN 0-471-93174-8

Microlocal analysis has permitted some substantial progress in spectral theory, and one of the most important contributions has been the clarification of the link between closed geodesics and the spectrum of the Laplacian, via the so-called Poisson relation [DG, C], in the case of a compact Riemannian manifold without boundary. To get an even wider historical perspective for the present book, we may add that this is quite natural, in view of the important role of WKB-techniques in microlocal analysis and the long history of such techniques in quantum mechanics. We may further recall that another area of WKB-techniques is geometrical optics.

The present book is devoted to (rigorous) spectral (and scattering) theory for the Laplacian (with suitable boundary conditions) on some domain Ω in $\mathbf{R}^n$ with smooth boundary, and particular attention is given to inverse problems, which is how to recover Ω or some associated geometrical quantities from spectral (or scattering) data. The authors follow the (classical) idea of studying the corresponding mixed problem for the wave equation on $\mathbf{R} \times \Omega$. Using some known results on the propagation of singularities for the solutions of the mixed problem for the wave-equation, the authors establish a general version of the Poisson relation for bounded Ω, saying roughly that the singularities of the trace of the wave-group are contained in the set of periods of all periodic rays, which obey (as much as possible) the optical laws of reflection. (We are here excluding the more subtle phenomena of diffraction, associated to analytic singularities, and we use the term wave group for the fundamental solution of the mixed problem, when it is viewed as a unitary one-parameter group.) Along a ray which reflects only transversally, it is fairly easy to understand completely the structure of the singularities of the solutions of the wave equation and hence also the corresponding contribution to the trace of the wave group. In general, however, rays may be tangential to the boundary and even glide along its concave parts, and, for such rays, little is known beyond the general theorem on the propagation of singularities already mentioned.

From the point of view of inverse problems, it is of interest to know when the Poisson relation becomes an equality or at least to know as many periods as possible which are in the singular support of the trace of the wave group. One also wants to know as explicitly as possible the shape of the corresponding singularities. Indeed, since the wave-trace depends only on the spectrum of the Laplacian, one could then recover periods of periodic rays and associated geometrical quantities, for instance, related to the corresponding Poincaré map.

The authors establish equality in the Poisson relation for generic bounded domains in $\mathbf{R}^2$ and for generic strictly convex domains in higher dimensions. In proving these results, some geometrical difficulties appear already with the transversally reflected rays. The first difficulty is that if the associated Poincaré map is degenerate, then there is no explicit formula for the corresponding contribution to the wave-trace. The second difficulty is that different transversally reflected rays with the same period may cancel each other's contributions to the trace. The geometric work (to which more than one-half of the book is devoted) is then to show that generically this type of difficulties do not occur, and this is done by a fairly systematic machinery, using the so-called multi-jet transversality theorem. These techniques are of independent interest.

A substantial part of the book is devoted to the analogous problems and results in scattering theory, in the case when Ω is a connected open set such that $\mathbf{R}^n\backslash\Omega$ is bounded. The scattering operator then plays a role analogous to the wave trace, and instead of periodic rays, one then considers so-called (ω, θ)-rays, which are unbounded rays having the direction ω on the incoming part and the direction θ on the outgoing part. Instead of the period we then have the notion of time delay. For these analogous quantities, we have a new Poisson relation, and again one is interested in when this relation is more explicit and when it is an equality. The authors establish many results in this direction, and in particular they explain how to recover geometric information about the obstacle from the singularities of the scattering operator. The results are particularly complete and interesting in the case of several strictly convex obstacles.

This book illustrates in one particular area the fascinating interplay between geometry and analysis. It is carefully written and contains some short but clear introductions both to microlocal analysis and to the geometric transversality techniques that are used. Hence, despite the fact that it deals with recent results, it should be of interest to a broader mathematical audience, including analysts, geometers, and advanced graduate students in those areas. The book does not pretend to cover all interesting aspects on spectral and scattering theory for boundary value problems nor on inverse problems (which would have been impossible), and one or two fundamental results are used without proof. Here are some further references:

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