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Ergodic theorems for group actions, by Arkady Tempelman. Kluwer Academic Publishers, Dordrecht, 1992, xviii+399 pp., $165.00. ISBN 0-7923-1717-3

The prototypes of the ergodic theorems treated in this book are the following two classical results.

The pointwise ergodic theorem [G. D. Birkhoff, 1931]. *Let T be a measurable measure-preserving transformation of a σ-finite measure space $(\Omega, \mathcal{F}, m)$ and $f \in L^p = L^p(\Omega, \mathcal{F}, m)$ for some $p \geq 1$. Then for m-almost all $\omega \in \Omega$ the limit*

$$(1) \qquad \overline{f}(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega)$$

exists and is T-invariant.

The mean ergodic theorem [J. von Neumann, 1932]. *Let U be a linear isometry of a Hilbert space H and $h \in H$. Then the limit*

$$(2) \qquad \overline{h} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} U^k h$$

exists (in the Hilbert space norm) and is U-invariant. In particular, setting $H = L^2$ and $Uf = f \circ T$, it follows that, for $f \in L^2$, the convergence in (1) also holds in the L^2-sense.

The adjective "ergodic", which may be circumscribed as "concerning the path of a phase point on an energy surface", hints at the origins in statistical mechanics. Here T describes the time evolution (in one time unit) on the phase space Ω of a mechanical system, m is the T-invariant "microcanonical" probability measure on a surface of constant energy, and one is interested in the question of whether the "asymptotic time average" (1) of an "observable" $f: \Omega \to \mathbb{R}$ exists and coincides with the "ensemble mean" $\int f\,dm$. By the theorems, the answer is affirmative when m and T are ergodic, in that each T-invariant function is constant m-almost everywhere. This last condition is a modern version of Boltzmann's famous ergodic hypothesis; its validity for systems of physical interest is still a major open problem.

Perhaps even more important than this application to statistical mechanics is the fact that the ergodic theorems above imply a general version of the law of large numbers. Indeed, let $\Omega = \mathbb{R}^{\mathbb{N}}$, T be the left-shift of coordinates, and $f: \Omega \to \mathbb{R}$ be the projection onto the first coordinate, so that $f \circ T^k$ is the projection onto the $(k+1)$th coordinate. The sequence $(f \circ T^k)_{k \geq 0}$ is then the canonical model for a sequence $(Z_n)_{n\geq 1}$ of real random variables, and the T-invariance of the associated probability measure m on Ω just means that $(Z_n)_{n\geq 1}$ is stationary, i.e., invariant (in distribution) under time shifts. It thus follows from (1) that, for each stationary sequence $(Z_n)_{n\geq 1}$ of integrable random variables, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Z_n$$

exists with probability 1.

In stochastic models of spatial phenomena one has to deal with families $(Z_x)_{x\in X}$ of random variables indexed by the elements of a group X like $\mathbb{Z}^d$ or $\mathbb{R}^d$. Such a family is called a random field on X, and it is called homogeneous if its distribution is invariant under translations of X. It is natural to ask for extensions of the law of large numbers to homogeneous random fields. Thus, one would like to replace the action of the semigroup $(T^k)_{k\geq 0}$ occurring in Birkhoff's theorem by the (right) action of an arbitrary group or semigroup X on $(\Omega, \mathscr{F}, m)$, i.e., a family $(T_x)_{x\in X}$ of measurable m-preserving transformations of Ω such that $T_{x_2}\circ T_{x_1} = T_{x_1x_2}$ for all $x_1, x_2 \in X$. Similarly, one would like to consider (right) isometric representations of X in a Hilbert space H, i.e., families $(U_x)_{x\in X}$ of linear isometries of H satisfying $U_{x_2}\circ U_{x_1} = U_{x_1x_2}$ for all $x_1, x_2 \in X$. This is the central theme of Tempelman's monograph. The basic question may be stated as follows:

For which measurable groups or semigroups X with a translation invariant measure μ and for which nets $(A_i)_{i\in I}$ in X is it true that:

(a) for every action $(T_x)_{x\in X}$ of X on a σ-finite measure space $(\Omega, \mathscr{F}, m)$ and each $f \in L^p$ with $p \geq 1$, the limit

$$\lim_{i\in I} \mu(A_i)^{-1} \int_{A_i} f(T_x\omega)\mu(dx) \tag{3}$$

exists for m-almost every $\omega \in \Omega$, and

(b) for every isometric representation $(U_x)_{x\in X}$ of X in a Hilbert space H and each $h \in H$,

$$\lim_{i\in I} \mu(A_i)^{-1} \int_{A_i} U_x h\ \mu(dx) \tag{4}$$

exists in the norm of H?

A net $(A_i)_{i\in I}$ in X satisfying (a) resp. (b) will be called *pointwise* resp. *mean averaging.*

Birkhoff (1931) already considered the case $X = \mathbb{R}_+$ (with Lebesgue measure μ) and proved that the standard net $([0, t])_{t\in\mathbb{R}_+}$ is pointwise averaging. The multiparameter case was treated by N. Wiener (1939) and N. Dunford (1939). They showed that unboundedly increasing sequences of cubes or balls in $X = \mathbb{R}^d$ or $\mathbb{Z}^d$ are both pointwise and mean averaging. Considering this result, one may restate the basic question above as follows:

(i) What are the essential geometric features of mean or pointwise averaging nets of sets?
(ii) Which semigroups X admit a net or a sequence of sets exhibiting these features?

One basic attribute of nets $(A_i)_{i\in I}$ of cubes or balls in $X = \mathbb{R}^d$ with volume $\mu(A_i)\to\infty$ is the so-called Følner property (with respect to Lebesgue measure μ): For each $x \in X$,

$$\lim_{i\in I} \mu(A_i \triangle A_i x)/\mu(A_i) = 0.$$

Here $A_i x$ stands for the translation of A_i by x, and $\triangle$ denotes the symmetric difference of sets. A locally compact group X which admits a net of compact Baire sets satisfying the Følner property with respect to (right) Haar measure μ is called *amenable.* It can be shown that X is amenable whenever it is abelian or compact or admits a composition series consisting of abelian or compact groups. On the other hand, a noncompact connected semisimple Lie group is not amenable. For an amenable group X, the mean ergodic problem has a simple solution: Each Følner net is mean averaging.

What can be said in the nonamenable case when no Følner net exists? Here are some typical results from the book. First, there is a special class of locally compact groups for which each net $(A_i)_{i\in I}$ with $0 < \mu(A_i) < \infty$ and $\mu(A_i) \to \infty$ is mean averaging; this class includes, for example, all noncompact connected almost simple Lie groups with finite center. For σ-compact connected locally compact groups X, one can take advantage of general structure theorems to construct mean averaging sequences. If X is not σ-compact and connected but nondiscrete, it can be shown, at least, that mean averaging sequences exist.

As for the pointwise ergodic problem, it can be shown by examples that a Følner net is not necessarily pointwise averaging. One needs an additional property called regularity. Unlike the Følner property which is not affected by translations of the sets A_i, the regularity condition imposes restrictions on their relative positions and on isolated parts far from the bulk. Typical examples of regular Følner nets in $\mathbb{R}^d$ are nets of bounded convex sets containing balls with diameters tending to infinity and nets of sets obtained by iterated stretching of a set of positive finite Lebesgue measure. A central result of the book implies that every linearly ordered increasing regular Følner net in an amenable group is pointwise averaging. It is interesting to note that the proof of this general theorem still follows the general device developed by Wiener (1939). For nonamenable but connected locally compact groups, it is possible to construct sequences of sets which are both mean and pointwise averaging.

Besides these results which are obtained by general methods, Tempelman's book also contains a discussion of more specific techniques which lead, for example, to the following recent result of R. L. Jones (1991): For each action $(T_x)_{x\in\mathbf{R}^d}$ of $\mathbb{R}^d$, $d \geq 3$, on a σ-finite measure space $(\Omega, \mathscr{F}, m)$ and each $f \in L^2$, the limit

$$\lim_{n\to\infty} \int f(T_x\omega)\sigma_n(dx)$$

exists for m- a.e. ω and in L^2-norm; here σ_n is the normalized surface measure on the sphere of radius n centered at the origin.

A further theme of Tempelman's book comes from the classical

Mean-value theorem [H. Bohr, 1925]. *Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is continuous and almost periodic, in that its translates $x \to f(x+y)$, $y \in \mathbb{R}$, form a relatively compact subset of the space $C(\mathbb{R}, \mathbb{C})$ (with the uniform norm). Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x+y)\,dx$$

exists uniformly in $y \in \mathbb{R}$ and does not depend on y.

From this result one is led to the following general question: For which (measurable) semigroups X, which continuous functions f from X into a Banach space B, and which nets $(\nu_i)_{i\in I}$ of probability measures on X, does the limit

$$\lim_{i\in I} \int f(yx)\nu_i(dx) \tag{5}$$

of the Bochner integrals $\int f(yx)\nu_i(dx) \in B$ exist (in the B-norm) uniformly in $y \in X$ and does not depend on y? In fact, this question extends the mean ergodic problem stated at (4). Indeed, the expression in (5) reduces to (4) if we set $B = H$, $\nu_i = \mu(\cdot \cap A_i)/\mu(A_i)$, y the identity of X (provided it exists), and $f\colon x \to U_x h$ the orbital function of $h \in H$ under the isometries U_x, $x \in X$. Note that the set of all translates of such an f belongs to a norm-ball in H which is well known to be weakly compact. In other words, the orbital functions occurring in the mean ergodic theorem and the concept of almost periodic functions admit a common generalization called weak almost periodicity: A function $f\colon X \to B$ is called *weakly almost periodic* if the set of its translates is relatively compact in the weak topology.

The question around (5) is approached in two stages:

(i) Which continuous functions $f\colon X \to B$ admit a sequence $(\nu_i)_{i\geq 1}$ of discrete probability measures on X satisfying (5)? This problem of "quasi-averageability" amounts to the question of finding invariant elements in the closed convex hull of all translates of f and can be solved by suitable fixed point theorems. For example, the answer is positive when X is a group and f is weakly almost periodic.

(ii) Which classes Q of continuous functions $f\colon X \to B$ admit a net $(\nu_i)_{i\in I}$ which is universally averaging, in that the limit (5) exists for all $f \in Q$? A typical result here is the following: If X is a connected σ-compact locally compact group, there exists an increasing sequence $(A_i)_{i\geq 1}$ of compact sets in X such that the measures $\nu_i = \mu(\cdot \cap A_i)/\mu(A_i)$ (μ being right Haar measure) form a universally averaging sequence for the class of all weakly almost periodic functions.

Some further topics discussed in Tempelman's book are: the ergodicity as well as the weak and strong mixing property for general group actions, ergodic theorems for homogeneous random fields on homogeneous spaces, ratio ergodic theorems, and local and global ergodic theorems for homogeneous random measures on locally compact groups. The final chapter contains some applications to information theory and statistical mechanics: (1) a Shannon-McMillan convergence theorem for the specific entropy per site of a homogeneous random field on a countable group—in fact, there are two distinct notions of specific entropy which coincide in classical cases but not in general; and (2) a variational characterization of homogeneous Gibbs random fields on countable groups in terms of their specific free energy—such Gibbs random fields are the stochastic model of a physical system of infinitely many interacting components in equilibrium.

The field of ergodic theorems for general group actions owes much to Tempelman's own contributions. His monograph (which is a largely extended version of a book published in 1986 in Russian) provides a comprehensive and impressive account of the subject. It also contains several new results and arguments. Unavoidably, there is some overlap with the standard work of Krengel [1], but the general scopes are different; roughly speaking, Krengel's book is focused on more specific situations. Besides bibliographical notes as usual and a long list of references, Tempelman's book contains an extensive appendix providing background information on various related subjects—this little encyclopedia is useful in its own right. On the negative side, besides misprints there are also omissions of important specifying words (like "weakly"), even in definitions, which require an alert reading. Also, a newcomer in the field may feel somewhat lost in technicalities because the author provides almost no guiding comments on motivations and intuitions. This, however, does not affect the eminent value of the book as an authoritative and complete reference work for ergodic theorists and other users of abstract ergodic theorems.

REFERENCES

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Combinatorics of train tracks, by R. C. Penner with J. L. Harer. Annals of Math. Stud., vol. 125, Princeton Univ. Press, Princeton, NJ, 1992, xii+216 pp., \$19.95. ISBN 0-691-02531-2

The theory of surfaces or Riemann surface theory, from its inception, has often been driven by computational and/or combinatorial questions. The roots of the theory lie in the study of algebraic integrals in the complex plane. Since that time, a significant part of surface theory has been devoted to the pursuit of combinatorial schemes to understand better some of the deeper structure, be it topological, geometric, algebraic, or analytic. The early work of Fagnano and later Euler, Abel, and Jacobi focused on the addition theorems for abelian integrals (see, e.g., Siegel [12]). In a different linguistic setting, we find this stream of ideas a still active part—and a spiritual foundation—of modern algebraic geometry. It is implicit in the constructions of Riemann, made concrete by Hurwitz, that a Riemann surface as a topological object may be defined by combinatorial data describing how to glue polygons together.

The complex analytic as well as the algebraic geometric approach to Riemann surface theory over most of the past one hundred years has been quite a bit less explicit. Emphasis was placed on broad general properties while interest in explicit algorithms lagged. Those of us in the complex analytic and topological wings of the Riemann surface community (as contrasted with those who study the same object under other names and guises) were treated to two jolts in the past two decades. The first came from Bill Thurston in the mid-1970s and underlies much of the substance of this review, while the second started about 1980 and came from particle physicists. The physicists actually wanted to do numerical computations, such as integration, in spaces of Riemann surfaces—a task for which we were quite unprepared.

The natural equivalence relation among Riemann surfaces is that of conformal or holomorphic equivalence. Not all topologically equivalent surfaces are conformally equivalent. The space of conformal equivalence classes of Riemann surfaces of a fixed finite topological type[1] is called the moduli space $\mathscr{M}$

[1]Here I am being sloppy by not distinguishing between punctures and bigger holes.