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BOOK REVIEW

The general theory of integration, by Ralph Henstock. Oxford Mathematical Monographs, Clarendon Press, Oxford, 1991, xi+262pp., $75.00. ISBN 0-19-853566-X

In elementary calculus courses we are usually successful in teaching students to evaluate an integral of a suitable function $f = F'$ on an interval $[a, b]$, by evaluating $F(b) - F(a)$, but we are often not very successful in connecting this type of integration with Riemann sums and their limits. During their junior/senior year, students who are studying mathematics seriously are then led through a more careful and exhaustive discussion of these ideas. However, they are informed that all of this is only tentative, since when they become graduate students they will replace the outmoded Riemann integral that they have just mastered with the Lebesgue integral. Of course, it is not *completely* replaced by this new integral, because there are certain notions, such as "improper integrals", that do not fall under this new umbrella and are still of considerable importance; moreover, almost all evaluations of integrals (whether Riemann or Lebesgue) are found by using the $F(b) - F(a)$ method, with a few minor variations. We tell our advanced undergraduates that we would like to introduce them to the Lebesgue integral but cannot do so since it requires a prior study of measure theory and/or topology and is "too advanced" for them at their present stage of mathematical study. Probably none of us is satisfied by this circuitous procedure.

Suppose that someone came up with an approach to the integral that simultaneously covered the integration of all functions that have antiderivatives, all functions that have Riemann integrals, all functions that have improper integrals, and all functions that have Lebesgue integrals. Moreover, suppose that the definition of this "superintegral" was only slightly more complicated than that of the Riemann integral, that its development required no study of measure theory, no study of topology, and that this integral had properties that correspond to the Monotone Convergence Theorem and the Lebesgue Dominated Convergence Theorem (among others).

If this mathematical miracle occurred, then wouldn't this new approach be immediately adopted, at least at the junior/senior level course, and quickly worked into the calculus level? The answer is a resounding: No!

Proof. In fact, such an integral has already been developed and has been around for some time, but its existence has remained largely unknown (except to readers of the *Real Analysis Exchange*) and it has had very little, if any, educational impact (known to this reviewer).

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The reader of this review may well be dubious of the above remarks. If, in fact, the definition of this superintegral is so simple, then what is it? Here goes, but first a couple of definitions will be convenient. (We will confine our attention to a compact interval $[a, b]$ for simplicity.) A *tagged division* of $[a, b]$ is a division (= partition) of $[a, b]$ given by a finite ordered set $a = x_0 < x_1 < \cdots < x_n = b$ of points, together with a collection of *tags* z_i such that $x_{i-1} \leq z_i \leq x_i$ for $i = 1, \ldots, n$. We denote a tagged division by $D(x_i, z_i)$ and the corresponding Riemann sum by

$$S(D(x_i, z_i)) := \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}) \,.$$

A *gauge* on $[a, b]$ is a function δ defined on $[a, b]$ such that $\delta(x) > 0$ for all $x \in [a, b]$. An important example of a gauge is a constant function. If δ is any gauge on $[a, b]$, we say that a tagged division $D(x_i, z_i)$ is δ*-fine* in case that $[x_{i-1}, x_i] \subseteq [z_i - \delta(z_i), z_i + \delta(z_i)]$; that is, in case $z_i - \delta(z_i) \leq x_{i-1} \leq z_i \leq x_i \leq z_i + \delta(z_i)$ for all $i = 1, 2, \ldots, n$. Finally, we say that the number A is an HK*-integral* of f if, for every $\varepsilon > 0$, there exists a gauge δ_ε such that if $D(x_i, z_i)$ is any tagged division of $[a, b]$ that is δ_ε-fine, then we have

$$|S(D(x_i, z_i)) - A| < \varepsilon \,.$$

It is easy to show that the HK-integral of a function is uniquely defined when it exists and that a function is Riemann integrable if and only if the gauge δ_ε can be chosen to be constant.

At first glance the above seems to be nothing particularly new. To see that the HK-integral "catches new fish", consider the Dirichlet function $g(x) := 0$ when x is irrational and $g(x) := 1$ when x is rational on $[0, 1]$. Let $(r_1, r_2, \ldots)$ be an enumeration of the rational numbers in $[0, 1]$, and, for $\varepsilon > 0$, define the gauge δ_ε by $\delta_\varepsilon(z) := 1$ if z is irrational and $\delta_\varepsilon(r_i) := \varepsilon/2^{i+1}$, $i = 1, 2, \ldots$. Thus, for any tagged δ_ε-fine division, the subintervals with rational tags have total length less than ε, and those with irrational tags contribute 0 to the Riemann sum. Thus the HK-integral of the function g is 0. This same argument can be used to show that the characteristic function of any Lebesgue null set is HK-integrable with integral 0. In fact, it can be shown that every Lebesgue integrable function is HK-integrable with the same value.

However, the HK-integral also integrates certain functions that are not Lebesgue integrable. Indeed, the derivative of the function $F(x) := x^2(\sin x^{-2})$ for $x \in (0, 1]$ and $F(0) := 0$ is neither Riemann nor Lebesgue integrable, but it is HK-integrable with integral $F(1) - F(0)$.

Further, let f be defined on $[0, 1]$ by $f(0) := 0$ and $f(x) := x^{-1/2}$ for $x \in (0, 1]$. Then, it is a somewhat tricky exercise to show that, if $0 < \varepsilon < 1$ and if the gauge δ_ε is defined by $\delta_\varepsilon(0) := \varepsilon^2/16$ and $\delta_\varepsilon(z) := \varepsilon z^{3/2}/4$ for $z \in (0, 1]$, then f is HK-integrable with integral $F(1) - F(0) = 2$. (Note that this coincides with the value of the improper integral of $x^{-1/2}$ over the interval $(0, 1]$. It is also interesting to notice that the gauge δ_ε forces the tag of the first subinterval in any δ_ε-fine division to be the point 0, where $f(0) = 0$.) In a similar way one can show that the function $h(x) := x^{-1} \sin x$ is HK-integrable on $[1, \infty)$. Thus, the HK-integral is not an "absolute integral", in the sense that the absolute value of an integrable function is not necessarily integrable. Moreover, a function f is Lebesgue integrable if and only if both it and its absolute value are HK-integrable.

The integral defined above is called the HK-integral (in this review), since it was developed in the late 1950s by Henstock [2] and Kurzweil [6]. It has also been called the "generalized Riemann integral" or the "gauge integral"; we propose that it be renamed simply "*the* integral". By now the reader may be so curious to learn about this integral that he/she may be ready to order a copy of the book under review and get to work. However, the reviewer would advise that the interested reader build up strength before doing so. Indeed, Henstock [5] published an earlier book intended as an introduction to this theory which the reader may find useful, and Kurzweil [7] published a monograph giving an exposition of this integration theory on $\mathbf{R}^n$, with all of its notational complications. McLeod [9] and Lee [8] present expositions that are approachable, although the first uses an idiosyncratic notation that did not help the reviewer, and the latter has certain gaps and obscurities this reviewer found annoying. By far the most readable account of the elementary aspects of this theory was presented by DePree and Swartz [1], which is highly recommended as an introduction to the theory. (McShare [10] made a modification of the definition that gives precisely the Lebesgue integral.)

What, then, is the content of the book under review? It consists of an extremely general and abstract treatment of the author's ideas. His strategy has apparently been to provide a formulation that is so general that almost every known "integration theory" is included as a special case. As a result, it is not always easy to identify some of the results as having any connection with any integration theory. Among the topics treated are the underlying notions of "division systems" and "division spaces", limit theorems, connections with differentiation theory, finite Cartesian products of division spaces, integration in infinite-dimensional spaces (which makes contacts with the Wiener and Feynman integrals), and a very general formulation of Riesz representation-type theorems. Perhaps the most readable section in the book deals with a "short history of integration", which is documented by forty-eight pages of references to the literature.

In conclusion, the book is an impressive monument to a lifetime of research in integration theory. However, in all honesty the reviewer cannot conceal his sorrow that the writing is so abstract and so general as to be virtually impenetrable by a typical reader. To quote the author in his assessment [5, p. 67] of the work of Denjoy, "The theory is extremely complicated, and only a dedicated student could hope to understand all its details." This reviewer hopes that there may be dedicated students who will have a go at the present book; their labor will not be easy, but it may be very fruitful.

References

1. John DePree and Charles Swartz, *Introduction to real analysis*, Wiley, New York, 1988.
2. Ralph Henstock, *The efficiency of convergence factors for functions of a continuous real variable*, J. London Math. Soc (2) **30** (1955), 273–286.
3. ______, *Definitions of Riemann type of the variational integrals*, Proc. London Math. Soc. (3) **11** (1961), 402–418.
4. ______, *Theory of integration*, Butterworths, London, 1963.
5. ______, *Lectures on the theory of integration*, World Scientific Publishing Co., Singapore, 1988.
6. Jaroslav Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. **7** (1957), 418–446.
7. ______, *Nichtabsolut konvergente Integrale*, Teubner-Texte, vol. 26, Teubner, Leipzig, 1980.

8. Peng-Yee Lee, *Lanzhou lectures on Henstock integration*, World Scientific Publishing Co., Singapore, 1989.
9. Robert M. McLeod, *The generalized Riemann integral*, Carus Math. Monographs, vol. 20, Math. Assoc. America, Washington, DC, 1980.
10. E. J. McShane, *A unified theory of integration*, Amer. Math. Monthly **80** (1973), 349–359.
11. ______, *Unified integration*, Academic Press, New York, 1983.

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