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Multidimensional inverse scattering problems, Alexander G. Ramm. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 51, Longman Scientific & Technical, Harlow, 1992, 379 pages, $170.00. ISBN 0-582-05665-9

In recent years the field of differential equations has come to distinguish between two different types of problems: the *direct* and the *inverse*. Broadly speaking, in the direct problems a differential equation is given and a particular solution is sought from among a given class of functions; in the inverse problems, a solution is given and a particular differential equation is sought from among a given class of equations. This distinction is driven by the widespread use of differential equations in the world about us. If we know exactly the physical laws and the experimental setup of a particular experiment, then we can predict the outcome by resolving a direct problem. Comparison with the empirical outcome will confirm the theory or suggest a revision. But if we do not know exactly the physical laws or the experimental setup, then perhaps we can recover the missing details by carefully measuring the outcome and resolving an inverse problem. Direct problems have been with us since Newton, but inverse problems are newer; they have come to fruition only since the Second World War, in fields as diverse as quantum mechanics, radar, tomography, and geological surveying.

The first postwar discussion of a recognizable inverse problem, in 1946, is due to Borg [1], who was concerned with the problem of recovering the density function for a one-dimensional vibrating string from a knowledge of its eigenfrequencies and eigenweights. Shortly thereafter there arose a considerable interest in determining the shape of certain nuclear potentials in quantum mechanics from measurements obtained from the scattering of elementary particle wave functions by these potentials. In 1949 Levinson [2] showed that the potential function which scatters a one-dimensional particle is uniquely determined by the asymptotic phase of the particle wave function. In 1952 Jost and Kohn [3] gave a simple algorithm for constructing the potential function from the asymptotic phase. Meanwhile, in 1951 Gelfand and Levitan [4] produced a general method for recovering the potential function in the one-dimensional Schrödinger equation from the spectral data, and Marchenko [5] extended the method to include recovering the potential directly from the scattering data. All these attempts took on a renewed interest in the 1960s, when it was discovered by Gardner, Greene, Kruskal, and Miura [6] that the direct problem for the nonlinear Korteweg-deVries equation could be completely resolved by first resolving an associated inverse problem for the linear Schrödinger equation. The inverse problem for the Schrödinger equation then took on all the aspects of a thriving cottage industry.

The extension of these results to the more realistic and more interesting cases in higher dimensions has not come easily. The difficulties are all present in the prototype problem of the scattering in three dimensions of an elementary particle by a scalar potential. Mathematically, the problem can be briefly stated this way: Consider the time-independent Schrödinger equation

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) - q(\mathbf{x})u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \mathbf{R}^3.$$

This equation governs the scattering in three dimensions of the quantum mechanical wave function $u(\mathbf{x})$ by the potential $q(\mathbf{x})$. The relevant solution $u(\mathbf{x})$ satisfies the associated integral equation

$$u(\mathbf{x}, \mathbf{k}) = \exp(i\mathbf{x}\cdot\mathbf{k}) - \int_{\mathbf{R}^3} \frac{\exp(i|\mathbf{k}|\,|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} q(\mathbf{y})u(\mathbf{y}, \mathbf{k})\,d\mathbf{y}.$$

As $|\mathbf{x}| \to \infty$, this solution has the asymptotic form

$$u(\mathbf{x}, \mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}} - T(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{4\pi r} + o(1/r).$$

This form may be interpreted physically as consisting of an ingoing plane wave plus an outgoing spherical wave weighted by the "T-matrix"

$$T(\mathbf{k}', \mathbf{k}) = \frac{1}{4\pi}\int_{\mathbf{R}^3} e^{-i\mathbf{k}'\cdot\mathbf{y}} q(\mathbf{y})u(\mathbf{y}, \mathbf{k})\,d\mathbf{y},$$

which embodies the measurable scattering information. Here $\mathbf{k}$ is the ingoing plane wave vector and $\mathbf{k}'$ is the outgoing scattered wave vector, with $r = |\mathbf{x}|$ and $k = |\mathbf{k}| = |\mathbf{k}'|$. Writing $\theta = \mathbf{k}/|\mathbf{k}|$, $\theta' = \mathbf{k}'/|\mathbf{k}'|$, and

$$A(\theta', \theta, k) = T(\mathbf{k}', \mathbf{k}),$$

we can state the relevant three-dimensional scattering problems as follows:

The direct potential scattering problem: Given $q(\mathbf{x})$, find $A(\theta', \theta, k)$.

The inverse potential scattering problem: Given $A(\theta', \theta, k)$, find $q(\mathbf{x})$.

The inverse potential scattering problem breaks naturally into several pieces:

The uniqueness problem: Given $A(\theta', \theta, k)$, show that at most one potential $q(\mathbf{x})$ can give rise to $A(\theta', \theta, k)$.

The characterization problem: Given $A(\theta', \theta, k)$, find conditions which guarantee that at least one potential $q(\mathbf{x})$ can give rise to $A(\theta', \theta, k)$.

The existence problem: Given $A(\theta', \theta, k)$, show that at least one potential can give rise to $A(\theta', \theta, k)$.

The stability problem: Show that small changes in the data $A(\theta', \theta, k)$ result in small changes in the potential $q(\mathbf{x})$.

The reconstruction problem: Given $A(\theta', \theta, k)$, construct, analytically or numerically, at least one potential giving rise to $A(\theta', \theta, k)$.

The partial data problem: Given some portion of $A(\theta', \theta, k)$, construct at least one potential $q(\mathbf{x})$ giving rise to that portion of $A(\theta', \theta, k)$.

None of these problems is easy, and most are still open. The methods of Gelfand-Levitan and Marchenko do not lend themselves to an immediate transcription to three dimensions, and new considerations of a geometrical nature appear. For example, the three-dimensional inverse potential scattering problem is overdetermined, in that a five-parameter family of scattering data is being used to determine a three-parameter potential. This suggests that the five-parameter family of scattering data is subject to a two-parameter family relations. Numerous authors, including among many others Kay and Moses [7], Faddeev [8], and Newton [9], have tried to extend the remarkable results of Gelfand-Levitan and Marchenko to three-dimensional problems, with only partial success. The last word on the inverse potential scattering problem for the Schrödinger equation has not yet been heard.

Meanwhile, a similar circle of ideas arose in the fields of acoustic and electromagnetic radiation—first perhaps in military radar and sonar studies, where the game is to guess the shape of an unknown scattering object from an analysis of the scattering data. In 1959 Keller [10] showed that the radar backscattering data from a smooth convex scattering object determine its Gaussian curvature at the specular point. Nirenberg had already proved [11] that the Gaussian curvature of the surface of a smooth convex object as a function of the normal vector in turn determines uniquely the shape of the object (Minkowski's problem). In 1969 Lewis [12] showed that the backscattering data also determine the cross-sectional areas of the object, and Radon [13] had already proved that these cross-sectional areas also determine uniquely its shape (Radon's problem). Several authors have studied the related problem of using acoustic scattering data to determine the shape of a scattering inhomogeneity in the index of refraction of an otherwise homogeneous acoustic medium. In one dimension this problem can be brought to the form of a potential scattering problem by means of a suitable Liouville transformation and then resolved by the methods of Gelfand and Levitan. In three dimensions this inverse refraction scattering problem exhibits all the same difficulties as does the inverse potential scattering problem, but it also involves the additional problems raised by the existence of caustics.

The methods used here have also found application in such fields as geological surveys and medical and industrial diagnostics. The use of ultrasound transponders is in principle very similar to that of radar and sonar transponders, and the use of CAT and PET scanning devices lead directly to the inverse problem of determining the shape of a three-dimensional inhomogeneous medium from a partial knowledge of the forward scattering data. A nice review of these developments is found in Smith et al. [14]. In particular, they lead again to Radon's problem and to the use of the Radon transform. We can say with some assurance that the study of inverse problems will find increasing numbers of applications in our increasingly technological society and will be with us for some time to come.

I have taken some trouble to sketch these highlights of the rather diffuse history of inverse problems for differential equations only because the book under review here ignores them almost completely. In particular, not one of the fundamental papers listed below is cited in the bibliography. (I find it hard to imagine writing a book on inverse scattering without mentioning at least the papers of Gelfand and Levitan.) On the other hand, the relevant work of the author, consisting of five books and 106 papers, is cited in its entirety. This curious imbalance is unfortunate, because it makes it difficult for the interested reader to place in context and appreciate the considerable achievements of the author. To be sure, the reader is duly warned; the introduction states clearly that,

> This book presents in detail and essentially in a self-contained way the author's results and methods developed by the author for studying multidimensional inverse problems of scattering theory, geophysics and some other types of inverse problems, such as inverse spectral problems and inverse source problems. These methods are based on the new notion: completeness of the set of products of solutions to partial differential equations. This notion has been introduced and used systematically in the series of papers [R].

The book, however, is hardly self-contained; for some of the arguments and details of proof, the reader is directed to dig among the individual books and papers of the author. (For the proof of Lemmas 1, 2, and 3 of Section 3.2, for example, the reader is referred to the author's book *Scattering by obstacles* (D. Reidel, Dordrecht, 1986) without further guidance.) And the notion of completeness of products of solutions is hardly new; it was used already by Borg [1] and later by Deift and Trubowitz [15], among others.

The table of contents presents an assortment of topics in the subject of the title, guided only by the author's own particular interests. Chapter 1 provides a brief introduction, including the quotation above. Chapter 2 introduces the concept of inverse scattering problems and identifies some six of them, each with numerous variations. (These variations are all given code numbers, like IP12, and are referred to hereafter only by their code numbers, sorely testing the reader's powers of concentration.) Chapter 3 discusses in detail the uniqueness problem for potential, refractive, and boundary scattering, as well as scattering governed by Maxwell's equations and other hyperbolic equations, and includes the substantial original contributions of the author based on the method of completeness of products mentioned above. Chapter 4 gives analytical solutions to those few problems where such solutions are known and includes the author's very nice results for the boundary scattering problem. Chapter 5 considers various computational methods proposed by the author for resolving numerically the inverse potential, refraction, and boundary scattering problems. These are based either on expansion in spherical harmonics or on variational principles and are shown to converge under appropriate assumptions. Chapter 6 offers a digression into the realm of signal processing and includes algorithms for the inversion of the Laplace transform and the Radon transform with incomplete data.

Chapter 7 takes up the characterization and reconstruction problems for the case of three-dimensional potential scattering. The discussion includes a description of the reconstruction procedure of Newton, which is based on the method of Marchenko, and a complicated characterization of the scattering data, which is due to the author. Chapter 8 returns to the one-dimensional inverse problems to describe in detail the Gelfand-Levitan procedure and some of its consequences, and Chapter 9 briefly surveys other miscellaneous results, including inverse source problems, phase retrieval problems, and problems in integral geometry. Two appendices discuss low-frequency asymptotics and stability estimates for numerical procedures. At the end of the book there is a list of open problems, mostly highly technical, which have concerned the author, a table of the inverse problems discussed, a list of symbols, and some bibliographic notes.

At $170 it is hard to know what audience this book will command. It is perhaps best viewed as a summary and guide to the author's own considerable contributions to a difficult and important class of problems. The publisher declares that, "This book will be of use to a wide range of readers: research mathematicians, Ph.D. students, geophysicists, electrical engineers, material scientists, and mathematics instructors will find much new and useful material in this monograph."

All of those listed here will indeed find much new material. None of them will find it easy going. And utility, like beauty, will lie in the eye of the beholder.

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Symmetries and Laplacians: Introduction to harmonic analysis, group representations and applications, David Gurarie. North-Holland, Amsterdam, 1992, vii+453 pp., $128.50. ISBN 0444 886 125

Fourier analysis on abelian groups is of great importance in the modern world. It is also a very old idea—dating back to the ancient Babylonians. The Fast Fourier Transform or FFT has helped to revolutionize many things from weather prediction to home music systems. The FFT is a rewrite of the Fourier transform on a finite abelian group such as $\mathbb{Z}/n\mathbb{Z}$, the additive group of integers modulo n. It was used by Gauss in 1805 to compute the orbit of the asteroid Juno and popularized by Cooley and Tukey in 1965.

Fourier analysis on nonabelian groups such as the group $O(3)$ of rotations of 3-space and its quotient $O(3)/O(2)$—the sphere—is also quite old. Laplace and Legendre introduced expansions of functions in spherical harmonics in the 1780's in order to study gravitational theory. Such analysis is necessary for understanding any phenomena with spherical symmetry, for example, earthquakes, the hydrogen atom, and the solar corona. See the reviewer's book [18, Chapter 2]. Recently a fast algorithm was found by Driscoll and Healy [9] for computing the Fourier transform on the sphere.

Fourier analysis on nonabelian finite groups is of more recent origin, going back to Frobenius. There are applications to many things: elementary particles, crystals, statistics, error-correcting codes, geometry. Diaconis [7] gives many examples, for instance, in the analysis of data from a survey which asked people to rank where they want to live: city, suburbs, or country. Diaconis [7, p. 143]