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BOOK REVIEW

Symmetries and Laplacians: *Introduction to harmonic analysis, group representations and applications*, David Gurarie. North-Holland, Amsterdam, 1992, vii+453 pp., $128.50. ISBN 0444 886 125

Fourier analysis on abelian groups is of great importance in the modern world. It is also a very old idea—dating back to the ancient Babylonians. The Fast Fourier Transform or FFT has helped to revolutionize many things from weather prediction to home music systems. The FFT is a rewrite of the Fourier transform on a finite abelian group such as $\mathbb{Z}/n\mathbb{Z}$, the additive group of integers modulo n. It was used by Gauss in 1805 to compute the orbit of the asteroid Juno and popularized by Cooley and Tukey in 1965.

Fourier analysis on nonabelian groups such as the group $O(3)$ of rotations of 3-space and its quotient $O(3)/O(2)$—the sphere—is also quite old. Laplace and Legendre introduced expansions of functions in spherical harmonics in the 1780's in order to study gravitational theory. Such analysis is necessary for understanding any phenomena with spherical symmetry, for example, earthquakes, the hydrogen atom, and the solar corona. See the reviewer's book [18, Chapter 2]. Recently a fast algorithm was found by Driscoll and Healy [9] for computing the Fourier transform on the sphere.

Fourier analysis on nonabelian finite groups is of more recent origin, going back to Frobenius. There are applications to many things: elementary particles, crystals, statistics, error-correcting codes, geometry. Diaconis [7] gives many examples, for instance, in the analysis of data from a survey which asked people to rank where they want to live: city, suburbs, or country. Diaconis [7, p. 143] finds, using Fourier analysis on the group S_3 of permutations of three objects that "the best single predictor of f is what people rank last."

Much of the motivation of the delightful book by Gurarie comes from mathematical physics. One uses real abelian Fourier analysis, for example, to find the Green's functions or fundamental solutions of the Laplacian, heat, Schrödinger, wave equations, etc. See Gurarie, p. 92.

Fourier analysis on nonabelian continuous groups is still a very active field. Some of the major contributors in the century include H. Weyl, E. Cartan, Harish-Chandra, I. Gelfand, S. Helgason, and A. Selberg. The work being reviewed gives a good introduction to this topic. Let us summarize some of the results obtained by considering the finite group case. This provides a primer for the continuous theory without the prerequisites of measure theory and differential geometry. And the finite theory is of use whenever one wants to put anything on a computer.

Greenspan [10] argues that, since the universe is finite, it is more appropriate to use finite models than infinite ones. See also Nambu [16] for finite versions of the standard equations of mathematical physics such as the heat and wave equations considered by Gurarie in §2.4. See Terras [19] for more details on some of these remarks.

Suppose that G is a finite abelian group. Let $\mathbb{T}$ be the multiplicative group of complex numbers of norm one (the 1-dimensional torus). Then the *dual group* $\widehat{G} = \{\chi\colon G \to \mathbb{T} | \chi \text{ is a group homomorphism}\}$. One can show that $\widehat{G}$ is isomorphic to the original group. In the case that $G = \mathbb{Z}/n\mathbb{Z}$, under addition modulo n, the elements of $\widehat{G}$ have the form $e_a(x) = \exp(2\pi i a x/n)$, for $a, x \in \mathbb{Z}/n\mathbb{Z}$.

The *Fourier transform* of a complex-valued function f on G is

$$\hat{f}(\chi) = \sum_{x \in G} f(x)\chi(x) \quad \text{for } \chi \in \widehat{G}.$$

This transform has properties analogous to those of the Fourier transform on $\mathbb{R}$; for example, *Fourier inversion*:

$$f(x) = \frac{1}{|G|}\hat{\hat{f}}(-x).$$

The transformation has the property of changing *convolution*

$$(f * g)(x) = \sum_{y \in G} f(y) g(x - y)$$

into pointwise multiplication

$$(f * g)\hat{}(x) = \hat{f}(x) \cdot \hat{g}(x).$$

This implies that if M is a circulant matrix or the adjacency matrix of a cycle graph, then FMF^{-1} is diagonal. You can use this result to study random walks on such graphs. Diaconis [7] uses Fourier analysis to study questions such as how long it takes for a randomly walking creature to reach every vertex.

The simplest version of the *Heisenberg uncertainty principle* says that, if $\operatorname{supp} f = \{x \in G \mid f(x) \neq 0\}$, then

$$|\operatorname{supp} f\,| \cdot |\operatorname{supp} \hat{f}\,| \geq |G|.$$

See Donoho and Stark [8]. This implies that a function and its Fourier transform cannot both be highly localized. In quantum mechanics this translates to the statement that it is impossible to find a particle's position and momentum simultaneously. The uncertainty principle also shows its (ugly/beautiful) face in signal processing and medical imaging.

The *Selberg trace formula* or *Poisson summation formula* for the finite abelian group G says that, if H is a subgroup of G and f a function on G, then

$$\frac{1}{|H|}\sum_{h \in H} f(gh) = \frac{1}{|G|}\sum_{\chi \in (G/H)\hat{}} \hat{f}(\chi)\chi(g),$$

where

$$(G/H)\widehat{\ } = \{\chi \in \widehat{G} \mid \chi(h) = 1 \text{ for all } h \in H\}.$$

There are many consequences, for example, the Jessie MacWilliams identities in the theory of error-correcting codes (see MacWilliams and Sloane [14]) and Lechner's theorem on the extraction of prime implicants of switching functions (see Mukhopadhyay [15]). We should also note that the Poisson summation formula (and its generalization to the Selberg trace formula) is related to the method of images in mathematical physics (see Gurarie, pp. 104–109).

The nonabelian analogs of these results are similar. Suppose that G is a finite group. Then a (finite-dimensional) unitary *representation* π of G is a group homomorphism $\pi\colon G \to U(d_\pi)$. Here $U(n)$ is the unitary group of $n \times n$ complex matrices g such that $g^*g = I$, where g^* denotes the transpose conjugate of g. The unitary representation π is *irreducible* if it is not uniformly block diagonalizable. Two representations are *equivalent* if one can be obtained from the other by uniform change of basis. Let $\widehat{G}$ be a complete set of inequivalent irreducible unitary representations of G. If $f\colon G \to \mathbb{C}$, then the *Fourier transform* is

$$\hat{f}(\pi) = \sum_{g \in G} f(g)\pi(g).$$

Note that $\hat{f}(\pi)$ is a $d_\pi \times d_\pi$ matrix. The *Fourier inversion formula* says

$$f(x) = \frac{1}{|G|} \sum_{\pi \in \widehat{G}} d_\pi \mathrm{Tr}(\pi(x^{-1})\hat{f}(\pi)).$$

See Gurarie p. 131. Once more the convolution property holds. This has an immediate application to the study of random walks on graphs, for the Fourier transform can be used to block diagonalize the adjacency matrix of the graph, which is essentially the Laplacian for the graph. This is analogous to the fact that the Fourier transform on $\mathbb{R}^n$ diagonalizes constant coefficient differential operators such as the Laplacian (see Gurarie p. 92). For more examples from graph theory, see Gurarie (pp. 143–145), Diaconis [7, pp. 48–49], and Angel et al. [1, 2]. We find in these last papers that Fourier analysis is useful for studying Ramanujan graphs. A k-regular graph is *Ramanujan* if for all eigenvalues λ of the adjacency matrix with $|\lambda| \neq k$, we have $|\lambda| \leq 2\sqrt{k-1}$. Such graphs have possible applications in building communications networks because they have large expansion constants. See Bien [4], Fan Chung [6], and Lubotsky, Phillips, and Sarnak [13]. The name Ramanujan was attached to such graphs by Lubotsky, Phillips, and Sarnak [13] because the graphs they considered were proved to be Ramanujan by using the truth of the Ramanujan conjecture which bounds Fourier coefficients of holomorphic modular forms.

Next let us discuss the Selberg trace formula for a finite group G with subgroup H. This is taken from Arthur [3]. Suppose for simplicity that ω is a 1-dimensional representation of H. Let ρ be the *induced representation* of G; i.e., $\rho = \mathrm{Ind}_H^G \omega$, acting on the space

$$V_\omega = \{\phi\colon G \to \mathbb{C} \mid \phi(hx) = \omega(h)\phi(x), \text{ for all } x \in G,\ h \in H\}$$

by the right action $\rho(g)\phi(x) = \phi(xg)$, for $x, g \in G$. We extend the definition of the Fourier transform above to arbitrary representations, not just irreducible ones.

Then we find that for $\phi \in V_\omega$ we have

$$[\hat{f}(\rho)\phi](x) = \sum_{y \in G} f(y)\phi(xy) = \sum_{y \in H\backslash G} \sum_{h \in H} f(x^{-1}hy)\omega(h)\phi(y).$$

This means that the trace of the operator on V_ω is

$$\mathrm{Tr}(\hat{f}(\rho)) = \sum_{x \in H\backslash G} \sum_{h \in H} f(x^{-1}hx)\omega(h).$$

On the other hand, the representation ρ is a direct sum of integer multiples $m(\pi, \rho)$ of $\pi \in \widehat{G}$. So we find another formula for the trace which gives the *pre-trace formula*:

$$\sum_{\pi \in \widehat{G}} m(\pi, \rho)\mathrm{Tr}(\hat{f}(\pi)) = \sum_{x \in H\backslash G} \sum_{h \in H} f(x^{-1}hx)\omega(h).$$

This result can be used to prove the Frobenius reciprocity law and the Frobenius formula for the character of the induced representation. It clearly specializes to Poisson summation when G is abelian. Selberg went on to rewrite the right-hand side of this formula in terms of conjugacy classes in H. It is interesting to carry this out explicitly for $G = \mathrm{GL}(2, \mathbb{F}_q)$ and $H = \mathrm{GL}(2, \mathbb{F}_p)$, where $\mathbb{F}_q$ is the finite field with q elements and $q = p^r$. See Angel et al. [2]. The result looks rather similar to that obtained for real base field in Gurarie, §7.5, or see Terras [18, §3.7]. Of course, the explicit determination of the representations of G is required . Gurarie does the continuous version for $\mathrm{SL}(2, \mathbb{R})$ in Chapter 7. The finite field case is quite analogous. See Piatetski-Shapiro [17].

There are many applications of the Selberg trace formula for $\mathrm{SL}(2, \mathbb{R})$. Some of these can be found in Terras [18]. One is to derive the Weyl law for the asymptotics of eigenvalues of the non-Euclidean Laplacian for $L^2(\Gamma\backslash H)$, for Γ a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ such as $\mathrm{SL}(2, \mathbb{Z})$. See Gutzwiller [11] for a discussion from the point of view of mathematical physics. We should perhaps note that for higher rank Lie groups the theory becomes much more complicated. See Helgason [12] for a nice summary of Fourier analysis on symmetric spaces G/K. We attempted to do the case of $\mathrm{GL}(n, \mathbb{R})$ with applications to multivariate statistics and the theory of lattice packings of spheres, for example, in Terras [18]. Dorothy Andreoli [21] gives an explicit generalization of the Selberg trace formula to $\mathrm{SL}(3, \mathbb{R})$. See also Arthur [3].

It is also possible to create finite analogs of other results studied by Gurarie. For example, various authors have studied finite analogs of the Radon transform, for example, Bolker [5] and Velasquez [21]. The continuous version of the Radon transform is now important for computerized tomography, CAT scanners.

Here we have only managed to outline a small portion of the book for the finite group case of the subject matter of the book of Gurarie. Of course, the main examples in this book are continuous groups such as: the compact Lie groups, for which Gurarie derives Weyl's character formula and the Borel-Weil-Bott Theorem; the nilpotent groups for which the Kirillov orbit method is explained; and the semisimple groups such as $G = \mathrm{SL}(2, \mathbb{R})$ and quotients G/K such as the upper half plane and fundamental domains for discrete subgroups of G. The book ends with applications to mathematical physics, making use of Emmy Noether's theorem in the calculus of variations. The topics of Toda lattices, the hydrogen atom, and the Kepler problem are to be found here.

To summarize, this book is a wonderful addition to the literature. There are concrete examples everywhere. As the author says, "We never engage in 'abstract' studies for their own sake." The writing is elegant and enjoyable. Congratulations to the author.

However, I am not sure that the publisher should be congratulated for its pricing policy or its warning: "No responsibility is assumed by the publisher for any injury and or damage. . . from any use or operations of any methods, products, instructions or ideas contained in the material herein." Damage from the use of group representations?

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