

TOPOLOGICAL INVARIANCE OF INTERSECTION LATTICES OF ARRANGEMENTS IN $\mathbb{C}\mathbb{P}^2$

TAN JIANG AND STEPHEN S.-T. YAU

ABSTRACT. Let $\mathcal{A}^* = \{l_1, l_2, \dots, l_n\}$ be a line arrangement in $\mathbb{C}\mathbb{P}^2$, i.e., a collection of distinct lines in $\mathbb{C}\mathbb{P}^2$. Let $L(\mathcal{A}^*)$ be the set of all intersections of elements of \mathcal{A}^* partially ordered by $X \leq Y \Leftrightarrow Y \subseteq X$. Let $M(\mathcal{A}^*)$ be $\mathbb{C}\mathbb{P}^2 - \bigcup \mathcal{A}^*$ where $\bigcup \mathcal{A}^* = \bigcup \{l_i: 1 \leq i \leq n\}$. The central problem of the theory of arrangement of lines in $\mathbb{C}\mathbb{P}^2$ is the relationship between $M(\mathcal{A}^*)$ and $L(\mathcal{A}^*)$.

Main Theorem. *The topological type of $M(\mathcal{A}^*)$ determines $L(\mathcal{A}^*)$.*

As a corollary of this, we show that the algebra and homotopy type of $M(\mathcal{A}^*)$ do not determine the topological type of $M(\mathcal{A}^*)$.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in \mathbb{C}^3 . Let $M(\mathcal{A}) = \mathbb{C}^3 - \bigcup \{H_i: 1 \leq i \leq n\}$. There is a standard procedure in [8] or [10] for passing from arrangements of hyperplanes in \mathbb{C}^3 to arrangements of lines in $\mathbb{C}\mathbb{P}^2$. In fact, $M(\mathcal{A}) = M(\mathcal{A}^*) \times \mathbb{C}^*$. The intersection lattice $L(\mathcal{A})$ is the set of all intersections of elements of \mathcal{A} partially ordered by reversed inclusion. [9] shows $L(\mathcal{A})$ completely determines the cohomology ring $H^*(M(\mathcal{A}))$.

This result brings the relation between $L(\mathcal{A}^*)$ and $M(\mathcal{A}^*)$ into focus. An example question: Does $L(\mathcal{A}^*)$ determine the homotopy type, topological type, and diffeomorphic type of $M(\mathcal{A}^*)$? Conversely, do any latter invariants of $M(\mathcal{A}^*)$ determine $L(\mathcal{A}^*)$?

For a general class of projective arrangements in $\mathbb{C}\mathbb{P}^2$, we have shown $L(\mathcal{A}^*)$ determines the diffeomorphic type of $M(\mathcal{A}^*)$ [4, 5].

Falk introduced an algebraic invariant for $L(\mathcal{A}^*)$. For two particular projective arrangements in $\mathbb{C}\mathbb{P}^2$, he asked if they have isomorphic Orlik-Solomon algebras [1, 2]. Rose and Terao produced such an isomorphism [11, 10]. Then Falk showed the $M(\mathcal{A}^*)$ s in his example have the same homotopic type. In view of this example, one would like to ask whether $L(\mathcal{A}^*)$ is determined by the topological type of $M(\mathcal{A}^*)$. The purpose of this note is to announce an affirmative answer to the above question.

Let us restate the main theorem more clearly:

Main Theorem. *Let \mathcal{A}_1^* and \mathcal{A}_2^* be two projective line arrangements in $\mathbb{C}\mathbb{P}^2$. If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then $L(\mathcal{A}_1^*)$ is isomorphic to $L(\mathcal{A}_2^*)$.*

In view of the results of Rose-Terao [11] and Falk [3], the following statement follows immediately from the main theorem.

1991 *Mathematics Subject Classification.* Primary 05B35, 14B05, 14F45, 57N20.

Received by the editors May 5, 1992 and, in revised form, December 31, 1992

Research was partially supported by NSF grant DMS8822727

Corollary. *There exist \mathcal{A}_1^* and \mathcal{A}_2^* , two projective line arrangements in $\mathbb{C}\mathbb{P}^2$ such that $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ have the same homotopy type and isomorphic cohomological algebra, but not the same topological type.*

Because of the main theorem, it makes the first question raised above more interesting. In fact, we believe that the following conjecture is true.

Conjecture. *For any projective line arrangement \mathcal{A}^* in $\mathbb{C}\mathbb{P}^2$, the topological type of $M(\mathcal{A}^*)$ is determined by $L(\mathcal{A}^*)$.*

In order to prove the main theorem, we have to separate arrangements in $\mathbb{C}\mathbb{P}^2$ into two categories. An arrangement in $\mathbb{C}\mathbb{P}^2$ is called *exceptional* if one of its lines has at most two intersection points. An arrangement in $\mathbb{C}\mathbb{P}^2$ is called *nonexceptional* if every line in the arrangement has at least three intersection points.

A *regular neighborhood* of an arrangement \mathcal{A}^* in $\mathbb{C}\mathbb{P}^2$ can be defined as follows: Choose a finite triangulation of $\mathbb{C}\mathbb{P}^2$ in which $\bigcup \mathcal{A}^*$ is a subcomplex. The closed star of $\bigcup \mathcal{A}^*$ in the second barycenter subdivision of this triangulation is then a regular neighborhood of \mathcal{A}^* .

The fundamental observation is the following lemma.

Lemma 1. *Let U_1 and U_2 be regular neighborhoods of \mathcal{A}_1^* and \mathcal{A}_2^* respectively. If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then ∂U_1 is homotopic equivalent to ∂U_2 .*

Let \mathcal{A}^* be an arrangement in $\mathbb{C}\mathbb{P}^2$. Suppose that \mathcal{A}^* has x_1, \dots, x_k ($k \geq 0$) as multiple intersection points (i.e., multiplicity $t(x_i) \geq 3$). By blowing up $\mathbb{C}\mathbb{P}^2$ at $\{x_1, \dots, x_k\}$, we get a set $\widetilde{\mathcal{A}}^*$ of lines in a blown-up surface $\widetilde{\mathbb{C}\mathbb{P}^2}$. $\widetilde{\mathcal{A}}^*$ is called an *associated arrangement* in $\widetilde{\mathbb{C}\mathbb{P}^2}$ induced by \mathcal{A}^* . Each pair of lines of $\widetilde{\mathcal{A}}^*$ intersects at most one point. Let $U(\widetilde{\mathcal{A}}^*)$ be a regular neighborhood of $\widetilde{\mathcal{A}}^*$ and $K(\widetilde{\mathcal{A}}^*) = \partial U(\widetilde{\mathcal{A}}^*)$. Thus $K(\widetilde{\mathcal{A}}^*)$ is a plumbed 3-manifold which is homeomorphic to $K(\mathcal{A}^*)$, the boundary of a regular neighborhood of \mathcal{A}^* in $\mathbb{C}\mathbb{P}^2$.

A class of 3-manifolds was classified by Waldhausen [12] in terms of *graphs* and *reduced graph structures* of 3-manifolds. We call these 3-manifolds classified in [12] as *Waldhausen graph manifolds*.

Lemma 2. *If \mathcal{A}^* is a nonexceptional arrangement in $\mathbb{C}\mathbb{P}^2$, then $K(\widetilde{\mathcal{A}}^*)$ is a Waldhausen graph manifold.*

We define a graph $G(\widetilde{\mathcal{A}}^*)$ of $\widetilde{\mathcal{A}}^*$ as follows. Let each vertex correspond to a line in $\widetilde{\mathcal{A}}^*$ with the weight of the self-intersection number of this line. Let each edge correspond to the intersection point of two lines in $\widetilde{\mathcal{A}}^*$.

We state some definitions and results derived from [12, 13]. Let M and N be compact orientable 3-manifolds. An isomorphism ψ of $\pi_1(N)$ onto $\pi_1(M)$ is said to *respect the peripheral structure* if for each boundary surface F in N there is a boundary surface G of M such that $\psi(i_*(\pi_1(F))) \subset R$ and R is conjugate in $\pi_1(M)$ to $i_*(\pi_1(G))$ where i_* denotes inclusion homomorphism.

Theorem 3 (cf. [13, (6.5)]). *If M and N are two Waldhausen graph manifolds and ψ is an isomorphism from $\pi_1(N)$ onto $\pi_1(M)$ which respect the peripheral structure and $H_1(M)$ is infinite, then there exists a homeomorphism from N to M which induces ψ .*

Theorem 4 (cf. [12, (9.4)]). *Two Waldhausen graph manifolds are homeomorphic if and only if the corresponding graphs are equivalent.*

Now suppose that \mathcal{A}_1^* and \mathcal{A}_2^* are two nonexceptional arrangements in \mathbb{CP}^2 and $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$. In view of Theorem 3 and Lemmas 1 and 2, we have that $K(\widetilde{\mathcal{A}}_1^*)$ is homeomorphic to $K(\widetilde{\mathcal{A}}_2^*)$. By Theorem 4 we conclude that there is an isomorphism from $L(\widetilde{\mathcal{A}}_1^*)$ to $L(\widetilde{\mathcal{A}}_2^*)$. This isomorphism also preserves weights (i.e., self-intersection number). So the main theorem follows from

Theorem 5. *Let \mathcal{A}_1^* and \mathcal{A}_2^* be two arrangements in \mathbb{CP}^2 . By blowing up their multiple points (of multiplicity ≥ 3), we obtain two associated arrangements $\widetilde{\mathcal{A}}_1^*$ and $\widetilde{\mathcal{A}}_2^*$ in some blown-up surfaces $\widetilde{\mathbb{CP}}^2$. Then there exists an isomorphism from $L(\mathcal{A}_1^*)$ onto $L(\mathcal{A}_2^*)$ which preserves weights if and only if there is an isomorphism from $L(\widetilde{\mathcal{A}}_1^*)$ onto $L(\widetilde{\mathcal{A}}_2^*)$.*

We next suppose that both \mathcal{A}_1^* and \mathcal{A}_2^* are exceptional. Write

$$(1) \quad \mathcal{A}_1^* = \{H_0, H_1, \dots, H_p, H_{p+1}, \dots, H_{p+q}\},$$

$$(2) \quad \mathcal{A}_2^* = \{H_0, G_1, \dots, G_s, G_{s+1}, \dots, G_{s+t}\}$$

where H_0 (respectively G_0) intersects with H_1, \dots, H_p (respectively G_1, \dots, G_s) at one point and interacts with H_{p+1}, \dots, H_{p+q} (respectively G_{s+1}, \dots, G_{s+t}) at another point. If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then the Orlik-Solomon algebras associated to \mathcal{A}_1 and \mathcal{A}_2 are isomorphic. It follows that $p+q = s+t$ and $pq = st$. So $L(\mathcal{A}_1)$ is isomorphic to $L(\mathcal{A}_2)$.

Finally, we assume that \mathcal{A}_1^* is exceptional, but \mathcal{A}_2^* is not. We need to show that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$. There are four subcases to consider.

Case a. \mathcal{A}_1^* consists of at most three lines. We need to observe only that the first betti number of $M(\mathcal{A})$ is precisely $|\mathcal{A}|$. So we have $b_1(M(\mathcal{A}_1)) \leq 3 < b_1(M(\mathcal{A}_2))$, and $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

Case b. \mathcal{A}_1^* is a pencil, and $|\mathcal{A}_1^*| \geq 4$. This follows immediately from the following two lemmas.

Lemma 6. *Let \mathcal{A}^* be an arrangement in \mathbb{CP}^2 . If \mathcal{A}^* is not a pencil (i.e., $\bigcap \mathcal{A}^* = \emptyset$) and $|\mathcal{A}^*| \geq 3$, then $b_3(M(\mathcal{A}))$, the third betti number of $M(\mathcal{A})$, is nonzero.*

Lemma 7. *Let \mathcal{A}^* be an arrangement in \mathbb{CP}^2 . If \mathcal{A}^* is a pencil (i.e., $\bigcap \mathcal{A}^*$ is a point), then $b_3(M(\mathcal{A}))$, the third betti number of $M(\mathcal{A}^*)$, is zero.*

Case c. \mathcal{A}_1^* consists of a pencil and a line in general position, and $|\mathcal{A}_1^*| \geq 4$ (see Figure 1).

By using Neumann's calculus of plumbing [7], one can show that $K(\mathcal{A}_1^*)$, the boundary of a regular neighborhood of \mathcal{A}_1^* , is a reduced graph manifold with reduced graph structure equal to empty set. It follows from Lemma 1 and Theorems 3 and 4 that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

Case d. $\mathcal{A}_1^* = \{H_0, H_1, \dots, H_p, H_{p+1}, \dots, H_{p+q}\}$ where $\bigcap_{i=0}^p H_i$ and $H_0 \cap (\bigcap_{i=p+1}^{p+q} H_i)$ are two different nonempty intersections, $p > 1$ and $q > 1$ (see Figure 2).

By blowing up the points $\bigcap_{i=1}^p H_i$ and $\bigcap_{j=p+1}^{p+q} H_j$, we get the following pictures (Figure 3) where E_1 and E_2 are exceptional lines.

FIGURE 1

FIGURE 2

Here the numbers in the parenthesis are the self-intersection numbers. We blow down H_0 to a point and get the following pictures (see Figures 4 and 5).

The graph manifold $K(\mathcal{A}_1^*)$ is then a Waldhausen graph manifold with the graph G . Notice that G has only zero weights, while the graph of $K(\mathcal{A}_2^*)$ has nonzero weight. Therefore, by Lemma 1 and Theorems 3 and 4 again, we know that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

FIGURE 3

FIGURE 4

REFERENCES

1. M. Falk, *The cohomology and fundamental group of a hyperplane complement*, Proceedings Iowa City Conference on Singularities, Contemp. Math. **90** (1989), 55–72.
2. ———, *On the algebra associated with a geometric lattice*, Adv. in Math. **80** (1990), 152–163.
3. ———, *Homotopy types of line arrangements*, preprint.
4. T. Jiang and S. S.-T. Yau, *Topological and differential structures of the complement of an arrangement of hyperplanes*, Amer. Math. Soc. Summer Institute on Differential Geometry, Proc. Sympos. Pure Math., vol. 54, part 2, Amer. Math. Soc., Providence, RI, 1993, pp.

FIGURE 5

- 337–358.
5. ———, *Diffeomorphic type of the complement of arrangement of hyperplanes*, submitted.
 6. ———, *Lattices and the topological structures of complements of arrangements in $\mathbb{C}\mathbb{P}^2$* , submitted.
 7. W. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), 299–344.
 8. P. Orlik, *Introduction to arrangements*, CBMS Regional Conf. Ser. in Math., vol. 72, Amer. Math. Soc., Providence, RI, 1989.
 9. P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), 167–189.
 10. P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, Berlin, Heidelberg, and New York, 1992.
 11. L. Rose and H. Terao, Private communication, 1988.
 12. F. Waldhausen, *Ein Klasse von 3-dimensionalen Mannigfaltigkeiten*, Invent. Math. **3** (1967), 308–333; **4** (1967), 87–117.
 13. ———, *On irreducible 3-manifolds that are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, BOX 4348, M/C
249, CHICAGO, ILLINOIS 60680

E-mail address, Tan Jiang: u25147@uicvm.bitnet

E-mail address, Stephen S.-T. Yau: u32790@uicvm.bitnet