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BOOK REVIEW

Representations and cohomology, II: *Cohomology of groups and modules*, by D. J. Benson. Cambridge University Press, London and New York, 1991, x + 278 pp., $59.95. ISBN 0-521-36135-4

Representations of finite groups over the complex numbers play a familiar role in many parts of mathematics as well as physics and chemistry. The classical theory of Frobenius and Schur, centering on group characters, has by now found its way into many algebra textbooks. Though the basic theory is well developed, the actual determination of irreducible representations (or characters) for particular groups remains challenging. For example, in the case of simple groups of Lie type, the characters have only recently been worked out in detail, largely through the efforts of George Lusztig (see [4]).

To probe characters (and group structure) more deeply, Richard Brauer emphasized the study of representations over a field k of prime characteristic p dividing the group order. The work of Brauer and a few other pioneers (notably J.A. Green) gradually attracted the interest of a wide circle of group theorists. The "modular" theory has become a central focus of research, offering many challenging problems and a diversity of possible approaches: see the surveys in [8, 10], for example, as well as the large treatises of Curtis and Reiner [5] and Feit [7]. For a selective introduction to the modular theory (of humane length), one might, however, want to start with the book of Alperin [1].

Over the complex numbers it is enough to determine the *irreducible* representations of a finite group G, which are realized within the group algebra (a semisimple ring). In the modular case the group algebra kG fails to be semisimple when the characteristic p of k divides $|G|$. Moreover, unless the Sylow p-subgroups of G happen to be cyclic, there will be indecomposable kG-modules of arbitrarily large dimension. To deal with these complications, one invokes techniques from ring theory and module theory—especially the general theory of representations of finite-dimensional algebras, with its exotic language of quivers and almost-split sequences. (Benson's first volume [3] and his earlier monograph [2] discuss many of these ideas in depth.)

Along with the module-theoretic viewpoint has come a pervasive use of the language and methods of homological algebra: projective and injective modules, resolutions and complexes, extensions, spectral sequences, derived categories, etc. In particular, the cohomology of a finite group becomes an essential object of study. (In the lead article of [8], Alperin goes so far as to declare that "cohomology is representation theory".) Dave Benson, along with his frequent collaborator Jon

Carlson, has done much to promote the creative interplay of topology, group cohomology, and modular representations. His Volume II provides an expert account of what is going on in this area. (For another expert approach to the cohomology of groups—with less emphasis on representation theory—see the monograph of Evens [6].)

Mac Lane's article [9] gives historical perspective on the development of group cohomology as a reflection of the cohomology of a topological space having the finite group as fundamental group. In his first couple of chapters, Benson gives a high-level survey (often without proofs) of both algebraic topology and the cohomology of groups, hoping to convince finite group theorists of the importance of the two-way connections involved. He then provides a concise review of spectral sequences, with emphasis on the spectral sequence of a group extension.

The cohomology groups of a finite group G (with coefficients in the trivial module k) fit together into a ring, which encodes in a mysterious way a lot of information about G (and is quite difficult to compute explicitly). A central result is the finite generation of the cohomology ring, for which both topological and algebraic proofs are given, following B. B. Venkov and Evens respectively. Other important tools in group cohomology are also introduced: the Bockstein homomorphism and the Steenrod operations.

But the centerpiece of Volume II is Chapter 5, dealing with "support varieties" attached to kG-modules. Say k is algebraically closed, of characteristic $p > 0$. The cohomology ring is only graded commutative (unless $p = 2$). But by discarding elements of odd degree when p is odd, one obtains a finitely generated commutative k-algebra $\mathrm{H}^\bullet(G, k)$. Its maximal ideal spectrum is a (homogeneous) affine variety V_G, the *support variety* of G, whose dimension is the Krull dimension of $\mathrm{H}^\bullet(G, k)$. (Since elements of odd degree in cohomology have square zero when p is odd, one has not sacrificed any geometric information.)

Here is a quick sketch of the further developments. (See Benson as well as Evens [6] and the survey articles by Alperin and Carlson in [8].) Around 1971 Quillen showed how to stratify V_G by locally closed subvarieties associated to conjugacy classes of elementary abelian subgroups (direct products of cyclic groups of prime order). This implies that $\dim V_G$ is the p-rank of G: the maximal rank of an elementary abelian p-subgroup of G. Following work of Alperin and Evens a decade later on the "complexity" of a kG-module M (an integer measuring the rate of growth of a minimal projective resolution), Carlson defined a homogeneous subvariety $V_G(M)$ of V_G of this dimension using an ideal in $\mathrm{H}^\bullet(G, k)$ naturally associated to M. For example, the projective modules are those of complexity 0, with $\{0\}$ as the support variety. In turn, Avrunin and Scott obtained a stratification of $V_G(M)$, generalizing Quillen's stratification of V_G, and showed that the support varieties for elementary abelian groups are the same as Carlson's "rank varieties" (which are defined without reference to cohomology and are reasonably computable). Nice formal properties emerge, for example:

$$V_G(M \oplus N) = V_G(M) \cup V_G(N),$$

$$V_G(M \otimes N) = V_G(M) \cap V_G(N),$$

the latter being consistent with the fact that tensoring with a projective module yields another projective module.

The stratification theorem says in effect that $V_G(M)$ is determined by the $V_E(M)$ and the maps on cohomology induced by conjugation and inclusion of elementary abelian p-subgroups E. This parallels to some extent the representation theory of G: for example, Chouinard's theorem that a kG-module is projective if and only if all its restrictions to elementary abelian subgroups are projective.

Support varieties have been calculated in some special cases. For example, in §5.13 of Volume II Benson works out the whole picture when G is a dihedral two-group and $p = 2$. With enough effort it is possible in this case to describe all the indecomposable kG-modules and then to pick out their support varieties in V_G, whose associated projective variety is a union of two projective lines meeting at a single point.

Without many such examples in hand it is hard to say precisely how far the support varieties illuminate the representation theory of G. It would be especially interesting to know more about the simple groups of Lie type. When p is the defining characteristic, one finds intriguing clues in the parallel theory of support varieties for the (restricted) Lie algebras of simple algebraic groups, as developed by Friedlander and Parshall and by Jantzen. There the nilpotent variety of the Lie algebra plays a key role.

Benson concludes his Volume II with two shorter chapters devoted to current research topics. Following Peter Webb, he discusses group actions on simplicial complexes and resulting generalized Steinberg modules (inspired by the case of groups of Lie type). Following Mark Ronan and Steve Smith, he describes G-equivariant local coefficient systems relative to subgroup complexes and resulting homology representations (again inspired by groups of Lie type).

Benson's exposition is locally clear and engaging, though (as he warns at the outset) the pace is brisk. Nonspecialists might at times wish for a global roadmap to help distinguish the main thoroughfares from the scenic byways, detours, and roads still under construction. But his books are ideally suited for a graduate student or practicing group theorist who wants to reach the current research frontier rapidly. There are occasional misprints and minor errors, of which the author has assembled a file. All in all, Benson has done an admirable job of drawing together the most important current ideas relating group representations to topology and group cohomology.

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